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Some types of Retractable and Compressible Modules

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بعض انواع من المقاسات المنضغطه و المنكمشه

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من قبل

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باشراف

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بسم الله الرحمن الرحيم

((قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ)).

صدق الله العظيم

سورة البقرة ، الآية ٣٢

الاهـداء

الى من أنار ومهد لي الطريق ومنحني شعاع الامل

رمز الوفاء والاخلاص أبي

الى من سهرت الليالي وروتني بحبها وحنانها وعطفها

رمز المحبه والعطاء والحنان أمي

الى من غمروني بحبهم وحنانهم

الحبيبات العزيزات أخواتي

الى نور عيني وفرحة عائلتي أخي

الى من لاينقطع دعائهم لي ابدا ازواج أخواتي

الى من تكاد بصماتها تظهر في كل مرحلة من

مراحل بحثي ويضيء شعاعها صفحات كتابتي

العالمة الفاضلة استاذتي المشرفة

د.ليلى

أهدي هذا الجهد البسيط

اسـراء حسـن

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Contents

Abstract

Let R be a commutative ring with unity, M and N be (left) unitary R-modules. Let $Hom_R(M,N)$ be the set of all R-homomorphisms from M to N. it is well-known that the properties of the R-module $Hom_R(M,N)$ are determined by the properties of R, M and N, and also some of the properties of M, N and R are determined by those of $Hom_R(M,N)$ so for this reason , the study of $Hom_R(M,N)$ attracted the attention of many researchers. Some special studies were appeared for $Hom_R(M,N)$ in case N is a non-zero submodule of M. by using the restriction that $Hom_R(M,N) \neq 0$ whenever N is a non-zero submodule of M, such module M is called retractable module. While when every non-zero submodule of M contains a copy of M, that means there exists a monomorphism in $Hom_R(M,N)$ whenever N is a non-zero submodule of M, M is called compressible module in this case. Clearly the class of compressible modules is contained properly in the class of retractable modules .Many studies about these notions were given.

Also some generalizations of these concepts were appeared such as essentially retractable modules, epi-retractable modules, small compressible and small retractable modules.

In this work, we shall give detailed study about small compressible modules and small retractable modules. Moreover we shall present other generalizations for compressible and retractable modules, namely, purely compressible modules, purely retractable modules, primely compressible modules and primely retractable modules.

Introduction

Let R be a commutative ring with unity, M and N be (left) R-modules. It is well-known that the R-module $Hom_R(M,N)$ which consists of all R-homomorphisms from M to N plays a central role in the study of many types of modules. By using the restriction that N is a non-zero submodule of M, some researchers were interested in studying modules for which $Hom_R(M,N) \neq 0$, such modules are called retractable modules. While in the case that $Hom_R(M,N)$ contains a monomorphism whenever N is a non-zero submodule of M, equivalently, every non-zero submodule of M contains a copy of M, such modules are called compressible modules. The concept compressible module was first used by J. M. Zelmanowitz in (1976) while the notion retractable module was first used by S. M. Khuri in (1979) since have many extensive studies were appeared about these two concepts, some of them were represent generalization for compressible and retractable modules, for instance, essentially retractable modules, Epi-retractable modules, see [1], [5], [7], [8], [20], [2١],[2٤],[2٥],[2٦],[3٥], [3٦],[٣٩], [4١] and [4٥].One of the generalizations of compressible and retractable modules was small compressible and small retractable modules which were introduced by H. K. Marhoon in (2014), we shall study these two generalizations in some details in chapter two of this thesis. By considering special classes of submodules of M, namely, pure submodules and prime submodules, we shall present and study the concepts of purely compressible modules, purely retractable modules, primely compressible modules and primely retractable modules.

The thesis contains four chapters. The first chapter represents preliminaries concerning compressible and retractable modules; this chapter consists of three sections. In section one, we present the notion of compressible modules with some of their known basic properties. In section two, we recall retractable modules with many properties of such modules. In section three, we gave many characterizations of retractable modules.

The following are some results of chapter one:

(1) Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is a retractable R-module, then every non-zero submodule of M is also retractable, see Proposition (1.2.٧).

(2) Let M be a torsion-free R-module. Then M is retractable if and only if M is dualizable, see Proposition (1.3.4).

(3) Let R be an integral domain. Then every finitely generated uniform R-module is retractable, see Proposition (1.3.1٢).

In chapter two, we shall give a detailed study for small compressible and small retractable modules, the chapter contains three sections. In the first section, we investigate the basic properties of small compressible modules. In the second section we shall concerned with the basic properties of small retractable modules. Some characterizations of small retractable modules are given in the third section. Moreover we introduce the concept of small epi-retractable modules with some of its basic properties.

Among other results in chapter two are the following:

(1) Let N be a proper submodule of an R-module M. If M/N is small compressible, then N is small prime submodule of M. see Theorem (2.1.17).

(2) Let M be an R-module in which every cyclic submodule of M is small in M. Let N be a small prime submodule of M such that $[N:M] \not\supseteq [K:M]$ for each submodule K of M containing N properly. Then M/N is small compressible. see Theorem (2.1.18).

(3) Let M be a fully invariant R-module such that $f(M)$ is a direct summand of M for each $f \in End_R(M)$. Then M is small retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $f(M)$ is small retractable, see Proposition (2.3.3).

(4) If R is a V-ring (or a von-Neumann regular ring), then every small projective R-module is small retractable, see Proposition (2.3.7).

Purely compressible and purely retractable modules are introduced and investigated in chapter three of this thesis. The chapter consists of four sections. Section one is devoted to the notion of purely compressible modules, we give a detailed study for these modules. We introduce a special type of purely compressible modules, in section two, namely purely critically compressible modules. The concept of purely retractable modules is presented and studied in section three; Finally, in section four, some properties and characterizations for purely retractable modules are given. Moreover we introduce the concept of purely epi- retractable modules with some of its basic properties.

We recall here some of the results of chapter three:

(1)Let M be a module having PSP and N be a proper submodule of an R-module M. If M/N is purely compressible, then N is purely prime submodule of M. see Proposition (3.1.26)

(2) Let M be a module such that every cyclic submodule of M is pure in M. If N is a proper purely prime submodule of M such that $[N:M] \not\supseteq [K:M]$ for all submodules K of M containing N properly. Then M/N is purely compressible. see Proposition (3.1.27).

(3) Let M be a faithful finitely generated multiplication R-module. Then M is purely compressible if and only if for each non-zero pure ideal I of R, $ann_M(I) = 0$, see Theorem (3.1.37).

(4) Let M be a faithful finitely generated multiplication R-module. Then M is purely compressible if and only if R is purely compressible ring, see Proposition (3.1.41).

(5) Let R be a ring in which every principle ideal is pure. Let M be a faithful finitely generated multiplication R-module such that every submodule of a pure submodule is also pure. Then the following statements are equivalent:

(i) M is purely compressible.

(ii) M is isomorphic to an R-module of the form A/P for some purely prime ideal P of R and an ideal A of R containing P properly.

(iii) M is isomorphic to a non-zero submodule of a finitely generated purely uniform, purely prime R-module., see Theorem (3.1.48)

(6) If N is a proper primely pure submodule of a module M such that $[N: M] \not\supseteq [K: M]$ for all submodules K of M containing N properly, then M/N is purely retractable, see Proposition (3.3.6)

(7) Every finitely presented module is purely retractable, see Corollary (3.4.6).

(8) Let M be a module such that every non-zero pure submodule of M contains a non-zero direct summand of M. Then M is purely retractable, see Proposition (3.4.8).

In the last chapter of this thesis, we introduce and study another characterization of compressible and retractable modules which are primely compressible and primely retractable modules, This chapter included five sections. In section one we introduce the concepts of generalized prime modules and generalized prime submodules as generalization for prime module and prime submodule. We establish some of their properties which are needed in the next sections of this chapter. In the second section, we give the concept primely compressible modules with some examples and basic properties of such modules are investigated. Section three is devoted for the concept of primely critically compressible modules. while section four contains the definition and many properties of primely retractable modules. Finally, in section five, some properties and characterizations for primely retractable modules are given. Moreover the notion of primely epi-retractable modules is presented with establishing some of its properties.

Among the results of chapter four are the following:

(1) Every primely compressible module is primely uniform, see Proposition (4.2.14)

(2) Let M be a faithful finitely generated multiplication R-module. If M is primely compressible, then for each non-zero prime ideal I of R,$ann_M(I) = 0$, see Theorem (4.2.15).

(3) Let R be a ring such that every non-zero principal ideal of R is prime. If M is a faithful finitely generated multiplication and $ann_M(I) = 0$ for each non-zero prime ideal I of R, then M is primely compressible, see Theorem (4.2.16).

(4) Let M be a faithful finitely generated multiplication R-module then M is primely compressible if and only if M is generalized prime, see Corollary (4.2.23)

(5) Let R be a ring in which every principal ideal is prime. Let M be a Z-regular faithful finitely generated multiplication R-module which satisfy

condition (*) such that a non-zero cyclic submodule of a direct summand of *M* is prime submodule of *M*. Then the following statements are equivalent:

(i) *M* is primely compressible.

(ii) *M* is isomorphic to an *R*-module of the form *A/P* for some generalized prime ideal *P* of *R* and an ideal *A* of *R* containing *P* properly.

(iii) *M* is isomorphic to a non-zero submodule of a finitely generated primely uniform, generalized prime *R*-module., see Theorem (4.2.27).

(6) Let *M* be a prime module. If *M* is primely critically compressible, then *M* is indecomposable but not conversely, see Proposition (4.3.8).

(7) Let *M* be a primely retractable quasi-Dedekind module. Then *M* is generalized prime and primely uniform,see Proposition (4.4.7).

(8) Every prime finitely generated projective module is primely retractable, see Corollary (4.5.5)

(9) Let *M* be a module such that every non-zero prime submodule of *M* contains a non-zero direct summand of *M*. Then *M* is primely retractable, see Proposition (4.5.6).

(10) Let *M* be a prime module such that every non-zero submodule of *M* contains a non-zero direct summand of *M*. if *M* is primely retractable, then *M* is retractable, see Proposition (4.5.7).

(11) Let *M* be a module satisfying (*). If *M* primely epi-retractable module, then every non-zero prime submodule of *M* is also primely epi-retractable, see Proposition (4.5.13).

Where a condition (*) means:

Let M be a module satisfying $\forall K \leq N \leq M$ if N is a prime submodule of M and K is a prime submodule of N, then K is a prime submodule of M.

Chapter One

Compressible and Retractable Modules

Introduction

This chapter represents a prelusion for the next chapters in our work. The chapter contains three sections. In the first section, we present the concept of compressible modules with some of its known basic properties and some related concepts which shall be needed later. The second section is devoted to the concept of retractable modules with many examples of such modules and many of its properties. In section three we give many characterizations of retractable modules.

1.1 Compressible Modules

We introduce in this section the concept of compressible modules with some of its basic properties. Also we recall the definitions of some concepts that are related to compressible modules.

"**Definition (1.1.1)[24]**

An R-module M is called ***compressible*** if M embedded in each of its non-zero submodule. That is for each non-zero submodule N of M, there exists a monomorphism $f: M \to N$".

A ring R is ***compressible*** if the R-module R is compressible.

Examples and Remarks (1.1.2)

(1) Z as Z-module is compressible.

(2) "A ring R is compressible if and only if R is an integral domain" [1].

Proof:

($\Rightarrow$) Let $a, b \in R$. Suppose that $ab = 0$ if $a \neq 0$, and $I = (a)$. Then I is a non-zero ideal of R and since R is compressible then there exists a monomorphism $f: R \rightarrow I$ .Let $f(1) = ra$ for some $0 \neq r \in R$. Then $f(b) = bf(1) = b(ra) = r(ab) = 0$, therefore $b = 0$. Hance R is an integral domain.

($\Leftarrow$) If R is an integral domain. let I be a non-zero ideal of R. Then there exists a non-zero element x in I. Define $f: R \rightarrow I$ by $f(r) = rx$ for all $r \in R$. Clearly f is a homomorphism and f is a monomorphism since R is an integral domain. Hence R is compressible.

(3) $Z_n$as a Z-module is not compressible module $\forall n \in Z_+, n > 1$.

(4) Q as a Z-module is not compressible since $Hom_Z(Q, Z) = 0$.

(5) Every simple R-module is compressible.

(6) Every non-zero submodule (direct summand) of a compressible module is also compressible.

(7)A homomorphic image of a compressible module need not be compressible. See examples (1) and (3).

(8) A direct sum of compressible modules need not be compressible in general. For instance, Z is a compressible Z-module but $Z \oplus Z$ is not compressible Z-module.

Proof:

Suppose that there exists $f: Z \oplus Z \to Z \oplus 0$ is a monomorphism. Then if $(a_1, b_1), (a_1, b_2) \in Z \oplus Z$ with $b_1 \neq b_2$, implies $(a_1, b_1) \neq (a_1, b_2)$ but $f(a_1, b_1) = f(a_1, b_2) = (a_1, 0)$ which is a contradiction.

The following concepts are needed in the next section.

"Definition (1.1.3)[17]

A proper ideal I of a ring R is called ***a prime ideal*** if $\forall a, b \in R$ with $ab \in I$ then $a \in I$ or $b \in I$".

"Definition (1.1.4)[20]

An R-module M is called ***prime*** if $ann_R(M) = ann_R(N)$ for each non-zero submodule N of M."

"Definition (1.1.5)[15]

A proper submodule N of an R-module M is called ***prime*** whenever $rx \in N$ for $r \in R$ and $x \in M$, then either $x \in N$ or $r \in [N :_R M]$, where $[N :_R M] = \{r \in R : rM \subseteq N\}$."

"Definition (1.1.6)[27]

A submodule N of an R-module M is called ***essential*** if for every non-zero submodule K of M, $N \cap K \neq (0)$."

"Definition (1.1.7)[27]

An R-module M is called ***uniform*** if $M \neq 0$ and every submodule of M is essential in M."

We recall in the following proposition, some properties of compressible modules, from [35], which will be needed in our work.

"**Proposition (1.1.8) [35, p.7, p.8]**

(1) Every compressible module is prime.

(2) A finitely generated module M is compressible if and only if M is uniform and prime module.

(3) Let M be an R-module. Then the following statements are equivalent:

(i) M is compressible.

(ii) M is isomorphic to an R-module of the form A/P for some prime ideal P of R and an ideal A of R containing P properly.

(iii) M is isomorphic to a non-zero submodule of a finitely generated uniform prime R-module".

"**Definition (1.1.9)[24]**

A compressible module M is called ***critically compressible*** if M cannot be embedded in any proper factor module M/N with N is a non-zero submodule of M".

Examples and Remarks (1.1.10)

(1) The Z-module Z is critically compressible.

(2) Every simple module is critically compressible.

(3) Q as a Z-module is not critically compressible.

(4) According to [26] the compressible and critically compressible modules are equivalent if R is a commutative ring.

If R is any ring we recall some of the following results:

Proposition (1.1.11)

A non-zero submodule of a critically compressible module is also a critically compressible.

Proof:

Let M be a critically compressible module and $0 \neq N \leq M$. Then N is compressible by Definition (1.1.9) and (Examples and Remarks (1.1.2.(6)). Let $0 \neq H \leq N$. Suppose that there exists a monomorphism, say $\alpha: N \to N/H$, and let $f: M \to N$ be a monomorphism. Hence the composition $M \xrightarrow{f} N \xrightarrow{\alpha} N/H \xrightarrow{i} M/H$ gives an embedding of M into M/H which is a contradiction. Therefore N is critically compressible.

" Definition (1.1.12)[8]

A ***partial endomorphism*** of an R-module M is a homomorphism from a submodule of M into M."

"Proposition (1.1.13)[26, proposition 1.1]

The following conditions are equivalent for a compressible module M

(1) M is critically compressible.

(2) Every non-zero partial endomorphism of M is a monomorphism."

1.2 Retractable Modules

We present in this section the concept of retractable modules with many examples and we investigate some of its properties.

"**Definition (1.2.1)[41]**

An R-module M is called ***retractable*** if $Hom(M,N) \neq 0$ for each non-zero submodule N of M."

"A ring R is called ***retractable*** if the R-module R is retractable."

Examples and Remarks (1.2.2)

(1)Every commutative ring with identity is a retractable.

Proof:

Let R be a commutative ring with identity and let I be a non-zero ideal of R. let $0 \neq a \in I$. Define $f: R \longrightarrow I$ by $f(r) = ra \, \forall r \in R$. Clearly f is a well-defined R-homomorphism. If $f = 0$,Then $f(r) = 0$ for all $r \in R$. So, $0 = f(1) = 1.a = 0$ which is a contradiction. Hence $Hom(R,I) \neq 0$.

(2) Z_n is a retractable Z-module for all positive integer $n > 1$.

(3) Every compressible module is retractable but not conversely, for instance Z_n as a Z-module is retractable but not compressible.

(4) Every integral domain is a retractable ring by (Examples and Remarks 1.1.2,(2)) and (3). However there is a retractable ring which is not an integral domain. For example Z_4 is a retractable ring but is not an integral domain.

(5) Q as a Z-module is not retractable since $Hom_Z(Q,Z) = 0$.

(6) Every semisimple (simple) module is retractable.

(7) If R is a semisimple ring, then every R-module is retractable.

(8) Let M be an R-module. Then M is a retractable R-module if and only if M is a retractable $R/ann(M)$-module.

Proof:

This follows from the fact that: N is an R-submodule of M if and only if N is an $\bar{R}$-submodule of M and $Hom_R(M,N) = Hom_{\bar{R}}(M,N), \overline{R} = R/ann(M)$.

(9) "For any R-module M, $R \oplus M$ is a retractable R-module."[39,p.71].In particular the Z-module $Z \oplus Q$ is retractable.

(10) "For any proper ideal I of a ring R, the R-module R/I is retractable." [36,p.306]

(11) "Z_{P^∞} as a Z-module is not retractable"[39,p.71].

(12) "Direct sum of any family of retractable modules is retractable." [36,proposition1.4,p.307]. In particular "An arbitrary direct sum of copies of M is retractable if and only if M is retractable." [42,proposition2.10,p.686].

In the following proposition we show that retractability is preserved under isomorphism.

<u>Proposition (1.2.3)</u>

Let M_1 and M_2 be two isomorphic R-modules. Then $M_1$is retractable if and only if M_2 is retractable.

Proof:

Assume that M_1 is retractable and let $\varphi : M_1 \to M_2$ be an isomorphism. Let $0 \neq N \leq M_2$. Then $0 \neq \varphi^{-1}(N) \leq M_1$. Put $K = \varphi^{-1}(N)$. Let $f: M_1 \to K$ be a non-zero homomorphism and let $g = \varphi|_K$ then $g: K \to M_2$ is a homomorphism and $g(k) = \varphi(\varphi^{-1}(N)) = N,$ hence $g: K \to N$ is a homomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$, then $h \in Hom(M_2, N)$.If $h = 0$,then $0 = g(f\left(\varphi^{-1}(M_2)\right) = g(f(M_1))$implies that $f(M_1) \subseteq Kerg \subseteq Ker\varphi = 0$.Thus $f(M_1) = 0$, which is a contradiction. Therefore $Hom_R\ (M_2, N) \neq 0$ which is what we wanted.

In order to give other applications of proposition (1.2.3) we need to recall: "An *R*-module is called ***free*** if it has a basis." [18].

Corollary (1.2.4)

If *R* is an integral domain, then every free *R*-module is retractable.

Proof:

Let *M* be a free *R*-module with basis $\{X_\lambda : \lambda \in \Lambda\}$. Then $M \simeq \bigoplus_{\lambda\in\Lambda} R_\lambda$ where $R_\lambda \simeq R \quad \forall\, \lambda \in \Lambda$ by [18,lemma 4.4.3,p.89]. By (Examples and Remarks1.2.2,(4)) R is retractable and hence $\bigoplus_{\lambda\in\Lambda} R_\lambda$ is retractable by (Examples and Remarks1.2.2,(12)). Therefore *M* is retractable by proposition (1.2.3).

"**Remark (1.2.5)**

A submodule of a retractable module is not necessary retractable in general."[36,p.306].

Consider the following example, which is recalled in [36,p.306] without any details, we explain it as follows:

Example (1.2.6)

Let $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\}$ where R be a commutative ring with identity. S is a ring with identity with respect to addition and multiplication of matrices. The non-zero ideals of S are:

$I_1 = S, I_2 = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\right\}, I_3 = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} : a, c \in R\right\}, I_4 =$
$\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R\right\}$ $or\ I_5 = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in R\}$.

In each of these cases one can easily define a non-zero homomorphism from S to I, which means that S is a retractable S-module.

Now, let $I = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\right\}$.we claim that I is not a retractable submodule of S.

Note that $I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent element and hence I is an idempotent ideal.

Let $J = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in R\}$. J is a subideal of I and $JI = 0$. Suppose that there is a homomorphism, say $f: I \to J$.

Then $f(I) = f(I^2) = f(I)I \subseteq JI = 0$ and hence $f(I) = 0$, that means $f = 0$,therefore $Hom(I, J) = 0$.Hence I is not retractable.

In the following proposition we prove that under certain condition, the submodule of a retractable module is also retractable.

First," recall that a ring is called ***Boolean ring*** in case each of its element is an idempotent" [17]

Proposition (1.2.7)

Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is a retractable R-module, then every non-zero submodule of M is also retractable.

Proof:

Let $0 \neq N \leq M$ and $0 \neq K \leq N$. Then $Hom_R(M,K) \neq 0$. Let $f: M \to K$ be a non-zero homomorphism. Hence $fi: N \to K$ is a homomorphism where $i: N \to M$ is the inclusion homomorphism. We claim that $fi \neq 0$, Suppose that $fi = 0$, then $(fi)(N) = 0 = f(N)$,so $N \subseteq Kerf$ and hence $K \subseteq Kerf$, which implies that $f(M) \subseteq Kerf$ therefore $f(f(M)) = 0$. Let $j: K \to M$ be the inclusion homomorphism. Then $jf \in End_R(M)$and $jf(M) = f(M)$ *but* $(jf)^2(M) = (jf)(jf)(M) = jf\big(f(M)\big) = j(f\big(f(M)\big) = j(0) = 0$, and $(jf)^2(M) = (jf)(M)$ since $End_R(M)$ is a Boolean ring.Hence $j\big(f(M)\big) = f(M) = 0$. Therefore $f = 0$ which is a contradiction, thus $fi \neq 0$, therefore N is retractable.

Now we recall a special case for a submodule of a retractable module is also retractable.

Proposition (1.2.8)

If R is an integral domain, then every non-zero ideal of R is also retractable.

Proof:

Let I be a non-zero ideal of R and let J be a non-zero subideal of I. Then $J^2 \neq 0$ (since R is an integral domain). But $J^2 \subseteq JI$. So $JI \neq 0$. Therefore there exists $r \in J$ such that $I \neq 0$. Define $f: I \to J$ by $f(a) = ra$ for all $a \in I$. Cleary, f is an R-homomorphism and $f \neq 0$, hence I is retractable.

To give an application of proposition (1.2.8), we recall that:

"An R-module M is ***quasi-Dedekind*** if and only if every non-zero $f \in End_R$ (M) is a monomorphism." [11,theorem 1.5,p.26].

"An R-module M is called ***dualizable***, if $Hom_R(M, R) \neq 0$." [18].

Corollary (1.2.9)

Let M be a faithful quasi-Dedekind dualizable R-module. Then M is retractable.

Proof:

M being faithful quasi-Dedekind dualizable gives R is an integral domain and M is isomorphic to an ideal of R, [11.corollary 1.8 and corollary 2.3]. Therefore $M \simeq I$ for some ideal I of R. By proposition (1.2.8), I is retractable and hence M is retractable by proposition (1.2.3).

Note (1.2.10)

The condition M is dualizable can not be dropped in corollary (1.2.9), for example Q as a Z-module is faithful and quasi-Dedekind but not dualizable and Q is not retractable.

Remark (1.2.11)

A direct summand of a retractable module need not be retractable in general.

For instance, in example (1.2.6), if we take $I = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\}$ and $K = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in R\}$, it is clear that $S = I \oplus K$, I is not retractable while S is retractable.

As another example: $Z \oplus Q$ is a retractable Z-module by (Examples and Remarks1.2.2,(9)), however Q is not retractable Z-module.

Remark (1.2.12)

An epimorphic image (a quotient module) of a retractable module is not necessary retractable in general. As it is shown in the following:

In example (1.2.6), S is a retractable S-module and $I = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\}$ is not a retractable submodule of S. Define $f: S \to I$ by $f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$. It can be easily checked that f is an epimorphism. On the other hand $Kerf = \left\{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in R\right\}$ and $S/Kerf \simeq I$. Hence $S/Kerf$ is not retractable by proposition (1.2.3).

As it was mentioned in (Examples and Remarks1.2.2,(3)) that compressible module is retractable and the converse need not be true in general, we recall in the following some partial converse:

"**Proposition (1.2.13)[45,proposition 1.2,p.3]**

Suppose that M is a retractable R-module. If every non-zero $f \in End(M)$ is a monomorphism, then every non-zero element of $Hom_R(M, N)$ is a monomorphism, for any non-zero submodule N of M. In particular, M is compressible.

"**Proposition (1.2.14)[21,proposition 1.2.10,p.33]**

A retractable quasi-Dedekind module is compressible."

Proposition (1.2.15)

If M is a retractable quasi-Dedekind R-module. Then M is isomorphic to an R-module of the form A/P for some prime ideal P of R and an ideal A of R containing P properly.

Proof:

By (1.2.14) M is compressible and by (1.1.8) the result follows.

"**Proposition (1.2.16)[45,proposition 1.3,p.3]**

Let M be a retractable R-module. Then the following statements are equivalent:

(1) M is critically compressible.

(2)Every non-zero partial endomorphism of M is monomorphism."

"**Definition (1.2.17)[27]**

Let M be an R-module, put $Z(M) = \{m \in M: ann_R(m) \leq_e R\}$, $Z(M)$ is a submodule of M it is called the ***singular submodule*** of M, M is called singular if $Z(M) = M$ and M is called nonsingular if $Z(M) = 0$."

"**Proposition (1.2.18)[45,proposition 1.7,p.4]**

A retractable nonsingular uniform module is critically compressible."

1.3 Some Characterizations of Retractable Modules

We present in this section some characterizations of retractable modules. Also we discuss some necessary or sufficient conditions for a module to be retractable.

"**Proposition (1.3.1)[42,p.685]**

An R-module M is retractable if and only if there exists $0 \neq \varphi \in End_R(M)$ such that $Im\ \varphi \subseteq N$ for each non-zero submodule N of M."

Proof:

(⇒) Suppose that M is retractable. Let $0 \neq N \leq M$. Then $Hom_R(M,N) \neq 0$. Let $f: M \to N$ be a non-zero homomorphism.let $\varphi = if$ where $i: N \to M$ be the inclusion homomorphism, so $\varphi \in End_R(M)$ and $\varphi \neq 0$ since $f \neq 0$ and i is a monomorphism. Clearly, $Im\ \varphi = f(M) \subseteq N$.

(⇐) To prove M is retractable. Let $0 \neq N \leq M$. By hypothesis there exists a non-zero homomorphism $\varphi: M \to M$ and $\varphi(M) \subseteq N$. Therefore $\varphi: M \to N$ is a non-zero homomorphism, that is $Hom_R(M,N) \neq 0$ hence M is retractable.

Now, we give the following characterization

Proposition (1.3.2)

An R-module M is retractable if and only if $Hom_R(M, Rx) \neq 0$ for all $0 \neq x \in M$.

Proof:

(⇒) Suppose that M is retractable and let $0 \neq x \in M$. Put $N = (x) = Rx$ Then $0 \neq N \leq M$ and $Hom_R(M,N) \neq 0$. Hence $Hom_R(M, Rx) \neq 0$.

($\Leftarrow$) To prove M is retractable. Let $0 \neq N \leq M$ and let $0 \neq x \in N$, by hypothesis, $Hom(M, Rx) \neq 0$. Let $f: M \to Rx$ be a non-zero homomorphism. Then $if: M \to N$ is a homomorphism where $i: Rx \to N$ is the inclusion homomorphism, clearly $if \neq 0$ since $f \neq 0$ and i is a monomorphism. Hence $Hom_R(M, N) \neq 0$ this completes the proof.

"Let R be an integral domain and M be an R-module. Let $T(M) = \{m \in M: rm = 0$ for some $0 \neq r \in R\}.T(M) = M$ is a submodule of M, M is called a ***torsion*** module if $T(M) = M$, and M is called ***torsion-free*** if $T(M) = 0$."[18]

<u>Proposition (1.3.3)</u>

If M is a torsion-free cyclic R-module, then M is retractable.

Proof:

Let $M = Rx$ for some $0 \neq x \in M$ and M is torsion-free Let $0 \neq m \in M$ Define $f: M \to Rm$ by $f(rx) = rm$ for all $r \in R$. If $rx = 0$ then $r = 0$ since M is torsion free and hence $rm = 0$, therefore f is well-defined. Clearly, f is a homomorphism. If f =0 implies $rm = 0$, $\forall r \in R$, hence $m = 0$ which is a contradiction so, $f \neq 0$ and $Hom(M, Rm) \neq 0$, $\forall 0 \neq m \in M$. Thus M is retractable by proposition (1.3.2).

In the following result, we show that in the class of torsion-free modules, retractability is equivalent to dualization.

<u>Proposition (1.3.4)</u>

Let M be a torsion-free R-module. Then M is retractable if and only if M is dualizable.

Proof:

($\Rightarrow$) Suppose that M is retractable. Let $0 \neq x \in M$.Then by proposition (1.3.2), $Hom_R(M, Rx) \neq 0$. Let $f: M \to Rx$ be a non-zero homomorphism. Define $g: Rx \to R$ by $g(rx) = r$ for each $r \in R$. It can easily checked that g is a well-defined monomorphism and hence $0 \neq gf \in Hom(M, R)$. Therefore M is dualizable.

($\Leftarrow$) Suppose that M is dualizable.Let $0 \neq f: M \to R$ be a homomorphism. Let $0 \neq x \in M$. Define $g: R \to Rx$ by $g(r) = rx$ for all $r \in R$, Clearly, g is a well-defined homomorphism and since M is torsion-free, g is a monomorphism, so, $0 \neq gf \in Hom(M, Rx)$ hence by proposition (1.3.2), M is retractable.

Remark (1.3.5)

The condition M is torsion-free in proposition (1.3.4), cannot be dropped. For example Z_n is a retractable Z-module but Z_n is not dualizable, in fact Z_n is a torsion Z-module.

It is well-known that every faithful prime module is torsion-free [3], so the following result is an immediate consequence of proposition (1.3.4).

Corollary (1.3.6)

Let M be a faithful prime R-module. Then M is retractable if and only if M is dualizable.

In order to give other applications of proposition (1.3.4), we need to recall the following:

"An R-module M is called ***torsionless***, if M can be embedded in a direct product of copies of R, equivalently, the natural homomorphism $\varphi: M \to M^{**}$

is a monomorphism, where φ is defined by $\big(\varphi(m)\big)(f) = f(m)\,, \forall m \in M, \forall f \in M^* = Hom(M,R)$." [44].

"An R-module M is called ***projective*** if for any epimorphism $f: A \to B$ (A and B are any two R-modules) and for any homomorphism $g: M \to B$, there exists a homomorphism $h: M \to A$ such that $fh = g$." [18]. That is the following diagram is commutative.

$$\begin{tikzcd}
& M \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\
A \arrow[r, "f"'] & B \arrow[r] & 0
\end{tikzcd}$$

"**Lemma (1.3.7)[44,p.144]**

(1) A torsionless module is dualizable.

(2) Every free (projective) module is torsionless.

(3) If R is an integral domain, then every torsionless R-module is torsion-free.

(4) If R is an integral domain and M is a finitely generated torsion-free R-module, then M is torsionless."

According to this lemma the following are consequence of proposition (1.3.4)

Corollary (1.3.8)

If R is an integral domain, then every torsionless R-module is retractable and the converse is not true in general.

Proof:

Let M be a torsionless R-module. Then M is dualizable and torsion-free (by Lemma (1.3.7,(1) and (3)) and hence by proposition (1.3.4), M is retractable.

For the converse, Z_n as a Z-module is retractable but not torsionless.

Corollary (1.3.9)

If R is an integral domain and M is a finitely generated torsion-free R-module, then M is retractable.

Proof:

M being finitely generated torsion-free gives M is torsionless (by lemma (1.3.7,(4)) and by corollary(1.3.8) M is retractable.

Remark (1.3.10)

The condition M is finitely generated in corollary (1.3.9) is necessary, for example Q as a Z-module is not retractable in fact Q is not finitely generated

Corollary (1.3.11)

If R is an integral domain and M is a free (projective) R-module, then M is retractable and the converse is not true in general.

Proof:

M is torsionless by lemma (1.3.7,(2)) and M is retractable by corollary(1.3.8).

For the converse, the Z-module Z_n is neither free nor projective, but it is retractable.

In the following result we show that the class of finitely generated uniform modules over an integral domain is contained properly in the class of retractable modules.

Proposition (1.3.12)

Let R be an integral domain. Then every finitely generated uniform R-module is retractable.

Proof:

Let M be a finitely generated uniform R-module. Then $M = Rx_1 + Rx_2 + \cdots + Rx_n$ where $x_i \in M$ $\forall i = 1,2,\ldots\ldots,n$.Let $0 \neq N \leq M$.Then $N \leq_e$ M and hence for each $i = 1,2,\ldots\ldots,n$ there exists $t_i \in R$, $t_i \neq 0$ and $0 \neq t_i x_i \in N$ [27]. Let $t = t_1 t_2 \ldots\ldots t_n$. Then $t \neq 0$ and $0 \neq tx_i \in N$ for each $i = 1,2,\ldots\ldots,n$.Now, for each $m \in M$,$m = \sum_{i=1}^{n} r_i x_i$ with $r_i \in R$ $\forall i = 1,2,\ldots\ldots,n$, and $tm = \sum_{i=1}^{n} t\,(r_i x_i) = \sum_{i=1}^{n} r_i (tx_i)$, Hence tm$\in N$ $\forall m \in M$. Define $f: M \to N$ by $f(m) = tm$ $\forall m \in M$, clearly f is a non-zero homomorphism, hence $Hom(M,N) \neq 0$ and therefore M is retractable.

For the converse, Z_6 as a Z-module is retractable but not uniform.

Remark (1.3.13)

The condition M is finitely generated in proposition (1.3.12) cannot be dropped, for example Q as a Z-module is uniform however it is not retractable.

In order to give some consequences of proposition (1.3.12), we have to recall the following:

"An R-module M is called ***injective*** if for any monomorphism $f: A \to B$ (A and B are any two R-modules) and for any homomorphism $g: A \to M$, there exists a homomorphism $h: B \to M$ such that $hf = g$."[18].

"An ***injective hull*** of an R-module M denoted by $E(M)$ is defined to be an injective essential extension of M" [27].

That is $E(M)$ is an injective R-module and $M \leq_e E(M)$.

"An R-module M is called ***quasi-injective*** if every homomorphism from every submodule N of M to M can be extended to an endomorphism of M that is the following diagram is commutative" [19].

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{N} \arrow[r, "i"] \arrow[d, "f"'] & \mathbf{M} \arrow[dl, dotted, "g"] \\ & \mathbf{M} & \end{tikzcd}$$

"A submodule N of an R-module M is called **closed** in M if N has no proper essential extension in M, that is if $N \leq_e K \leq M$ implies $K = N$".[27]

"An R-module M is called ***indecomposable*** if 0 and M are the only direct summands of M" [27].

<u>Corollary (1.3.14)</u>

Let R be an integral domain and M be a finitely generated R-module If $E(M)$ is indecomposable, then M is retractable.

Proof:

Since $E(M)$ is indecomposable, then according to [27,Exercises7,p.94] M is uniform and hence the result follows from proposition (1.3.12).

Corollary (1.3.15)

Let R be an integral domain and M be a finitely generated R-module which has only two closed submodules. Then M is retractable.

Proof:

Since M has exactly two closed submodules implies that M is uniform, [27,Exercises8,p.94] and by proposition (1.3.12), we get M is retractable

Corollary (1.3.16)

Let R be an integral domain and M be a finitely generated indecomposable quasi-injective R-module. Then M is retractable.

Proof:

M being indecomposable and quasi-injective implies that M is uniform, [27,Exercises10,p.94] and according to proposition (1.3.12) M is retractable

For the next result the following concept is needed:

"An R-module M is called ***quasi-projective*** if for each epimorphism $f: M \to A$ (A is any R-module) and for each homomorphism $g: M \to A$, there exists a homomorphism $h: M \to M$ such that $fh = g$." [19].

A necessary and sufficient condition for the quotient of a quasi-projective module to be retractable was presented in [7, lemma 2.1,p.38] without proof, we shall give its proof here.

Proposition (1.3.17)

Let M be quasi- projective R-module and N be a submodule of M. Then M/N is retractable if and only if for all submodules L of M containing N, the set $A_L = \{f \in End_R(M) \mid f(N) \subseteq N, f(M) \subseteq L \text{ and } f(M) \nsubseteq N\}$ is non-empty.

Proof:

(⇒) suppose that M/N is retractable. Let $L \leq M$ and $L \supseteq N$. Then $L/N \leq M/N$ and hence there exists a non-zero homomorphism, say $\alpha: M/N \rightarrow L/N$. Let $\beta = i\alpha\pi$ where $\pi: M \rightarrow M/N$ is the natural homomorphism and $i: L/N \rightarrow M/N$ is the inclusion homomorphism. Then $\beta: M \rightarrow M/N$ is a homomorphism. Now consider the following diagram:

$$\begin{tikzcd} & \mathbf{M} \arrow[dl, dotted, "f"'] \arrow[d, "\beta"] \\ \mathbf{M} \arrow[r, "\pi"'] & \mathbf{M/N} \end{tikzcd}$$

where the homomorphism f exists and make the diagram commutative because M is quasi-projective by hypothesis. Therefore $\pi f = \beta$.

We claim that $f \in A_L$. $f \in End_R(M)$ and $\beta(N) = i\alpha\pi(N) = i\alpha(N) = i(N) = N$. On the other hand $\beta(N) = \pi f(N) = f(N) + N$. Therefore $f(N) + N = N$ and hence $f(N) \subseteq N$.

Next, we prove $f(M) \subseteq L$. Let $x \in f(M)$, then $x = f(m)$ for some $m \in M$. $\beta(m) = \pi f(m) = \pi(x) = x + N = i\alpha\pi(m) = i\alpha(m + N) = \alpha(m + N) \in \frac{L}{N}$. Thus $\alpha(m + N) = l + N$ for some $l \in L$ therefore $x + N =$

$l + N$ gives $x - l = n$ for some $n \in N$. Hence $x = f(m) = l + n \in L$, so $f(M) \subseteq L$.

Suppose that $f(M) \subseteq N$.then $\pi f(M) = \bar{0} = N$ and hence $\beta(M) = \bar{0}$ so $i\alpha\pi(M) = \bar{0} = \alpha(M/N)$ implies that $\alpha = 0$ which is a contradiction. Therefore $f(M) \nsubseteq N$.So, $f \in A_L$ and hence A_L is non-empty.

($\Leftarrow$) Assume that A_L is non-empty. To prove M/N is retractable. Let L/N be a non-zero submodule of M/N.Then L is a submodule of M containing N and $L \neq N$. By hypothesis, there exists a homomorphism $f: M \to M$ such that $f(N) \subseteq N, f(M) \subseteq L$ and $f(M) \nsubseteq N$. Thus $f: M \to L$ is a homomorphism and hence f induces a homomorphism $\bar{f}: M/N \to L/N$ where $\bar{f}(m + N) = f(m) + N$ for each $m \in M$. $\bar{f} \neq 0$, for if $\bar{f} = 0$, then $\bar{f}\ (M/N) = \bar{0} = N$ and hence $f(M) + N = N$ that is $f(M) \subseteq N$ which is a contradiction. Therefore $0 \neq \bar{f} \in Hom_R(M/N,\mathrm{L}/N)$. Hence M/N is retractable.

It is well-known that every projective module is quasi-projective, thus the following is a consequence of proposition (1.3.17)

Corollary (1.3.18)

Let M be a projective R-module and N be a submodule of M. Then M/N is retractable if and only if $f \in End_R(M)$ such that $f(N) \subseteq N, f(M) \subseteq L$ and $f(M) \nsubseteq N$ for any submodule L of M containing M.

For other consequences of proposition (1.3.17) we need to recall the following:

"A submodule N of an R-module M is called ***invariant*** if $f(N) \subseteq N$ for all $f \in End_R(M)$.M is called ***fully invariant*** if every submodule of M is invariant". [31]

Corollary (1.3.19)

Let M be a quasi-projective (projective) R-module and N be invariant submodule of M. Then M/N is retractable if and only if there exists $f \in End_R(M)$ such that $f(M) \subseteq L$ and $f(M) \nsubseteq N$ for all submodule L of M containing N.

Corollary (1.3.20)

Let M be a fully invariant quasi-projective (projective) R-module and N be a submodule of M. Then M/N is retractable if and only if there exists $f \in End_R(M)$ such that $f(M) \subseteq L$ and $f(M) \nsubseteq N$ for all submodule L of M containing N.

Recall that "An R-module M is called ***multiplication*** if every submodule N of M is of the form IM for some ideal I of R" [٤0].

In the class of multiplication modules, there is a relation between the retractabity of the module and that of the ring, namely, we give the following result:

Proposition (1.3.21)

Let M be a faithful multiplication R-module. Then M is retractable.

Proof:

Let $0 \neq N \leq M$. Then $N = IM$ for some non-zero ideal I of R, and since R is a retractable ring by (Examples and Remarks (1.2.2,(1)) implies that $Hom(R, I) \neq 0$, let $f: R \to I$ be a non-zero homomorphism. Put $f(1) = a$ for some $a \in I$. Then $a \neq 0$. Define $g: M \to N$ by $g(x) = ax$ for all $x \in M$. Clearly, g is a well-defined homomorphism. Moreover $g \neq 0$ since M is faithful. Therefore $Hom(M, N) \neq 0$ and hence M is retractable.

Remark (1.3.22)

The condition M is multiplication in Proposition (1.3.21) in necessary, for instance, Q as a Z-module is not retractable and it is not multiplication.

Corollary (1.3.23)

Every every faithful cyclic R-module is retractable.

Chapter Two

Small Compressible and Small Retractable Modules

Introduction

In this chapter we present a detailed study for the concepts small compressible modules and small retractable modules. This chapter consists of three sections. In section one we study small compressible modules by investigating the basic properties of this type of modules. In the second section we recall and study the basic properties of small retractable modules. Next, in the third section we introduce some characterizations of small retractable modules; moreover we give the relationships between these modules and certain types of modules; also we give the concept of small epi-retractable module with some of its basic properties.

2.1. Small Compressible Modules

The concepts of small compressible and small critically compressible modules are introduced in this section and many of their basic properties are studied, moreover we give some characterizations of these concepts.

"**Definition (2.1.1)[18]**

A proper submodule N of an R-module M is called ***small submodule*** $(N \ll M)$ if for any submodule K of M with $N + K = M$ implies $K = M$. Equivalently, N is a small submodule of M if for every proper submodule K of M, $N + K \neq M$".

Examples (2.1.2)

(1) (0) is a small submodule of every module.

(2) (0) is the only small submodule of the *Z*-module *Z*.

(3) Every finitely generated submodule of the *Z*-module Q is small in *Q*.

(4) $(\bar{2})$ is a small submodule of the *Z*-module Z_4.

(5) $(\bar{3})$ is not a small submodule in the *Z*-module Z_6.

"**Definition (2.1.3)[21]**

An *R*-module *M* is called ***small compressible*** if *M* can be embedded in each of its non-zero small submodule.

Equivalently, *M* is small compressible if there exists a monomorphism from *M* into *N* whenever $0 \neq N \ll M$".

A ring *R* is called ***small compressible*** if the *R*-module *R* is small compressible. That is *R* can be embedded in any of its non-zero small ideal.

Examples and Remarks (2.1.4)

(1) Every compressible module is small compressible and the converse is not true in general, for instance Z_6 as a *Z*-module is not compressible but Z_6 is small compressible since (0) is the only small submodule of Z_6 .

(2) Let *M* be a small compressible module such that every submodule of *M* contains a non-zero small submodule of *M*, then *M* is compressible.

Proof:

Let $0 \neq N \leq M$. By hypothesis there exists a small submodule $0 \neq K \ll N$, then $K \ll M$ [22,proposition 1.1.3,p.11] since *M* is small compressible there exists $f: M \to K$ is a monomorphism, $if: M \to N$ is a monomorphism where $i: K \to N$ be the inclusion homomorphism, then *M* is compressible.

(3) The *Z*-module Q is not small compressible since $Z \ll Q$ and $Hom(Q, Z) = 0$.

(4) Z_4 as a *Z*-module is not small compressible, since $(\bar{2}) \ll Z_4$ but Z_4 cannot be embedded in $(\bar{2})$.

(5) If M is a hollow module (every proper submodule of M is small in *M*). Then *M* is small compressible if and only if *M* is compressible.

(6) Every simple module is small compressible but not conversely, since Z as a *Z*-module is small compressible but not simple.

(7) Each of the rings *Z* and Z_6 is a small compressible ring.

(8)A module *M* is small compressible if and only if *M* can be embedded in *Rx* for each $0 \neq x \in M$ and $Rx \ll M$.

Proof:

(⇒) Is obvious according to the definition (2.1.3).

(⇐) Let $0 \neq N \ll M$ and let $0 \neq x \in N$.Then $Rx \ll M$[18,Lemma 5.1.3,p.108]. By hypothesis there is a monomorphism say, $f: M \to Rx$ so, the composition $M \xrightarrow{f} Rx \xrightarrow{i} N$ is a monomorphism with $i: Rx \to N$ is the inclusion homomorphism. Hence *M* is small compressible.

(9)A small compressible module *M* is compressible if every cyclic submodule of *M* is small in *M*.

Proof:

Let $0 \neq N \leq M$ and $0 \neq x \in N$. Then by hypothesis $Rx \ll M$ so there is a monomorphism $f: M \to Rx$ and hence the composition $M \xrightarrow{f} Rx \xrightarrow{i} N$ is a monomorphism which implies that M is compressible.

(10) Let M be a module in which every cyclic submodule of M is small in M. Then M is compressible if and only if M is small compressible.

<u>Proposition (2.1.5)</u>

A small submodule of a small compressible module is also small compressible.

Proof:

Let M be a small compressible module and $0 \neq N \ll M$. Let $0 \neq K \ll N$. Then $K \ll M$[18,Lemma 5.1.3,p.108] . As M is small compressible implies there exists a monomorphism, say $f: M \to K$ and therefore $fi: N \to K$ is a monomorphism where $i: N \to M$ is the inclusion homomorphism. Hence N is small compressible.

<u>Proposition (2.1.6)</u>

A direct summand of a small compressible module is also small compressible.

Proof:

Let $M = A \oplus B$ be a small compressible module and let $0 \neq K \ll A$. Then $K \oplus 0 \ll M$ [22,proposition1.1.4,p.11] and hence there is a monomorphism say, $f: M \to K \oplus 0$ clearly $K \oplus 0 \simeq K$, so $f: M \to K$ is a monomorphism and the composition $A \xrightarrow{j_A} M \xrightarrow{f} K$ is a monomorphism where j_A is the injection of A into M. Therefore A is small compressible.

Proposition (2.1.7)

Let M_1 and M_2 be two isomorphic modules. Then M_1 is small compressible if and only if M_2 is small compressible.

Proof:

Assume that M_1 is small compressible and let $\varphi : M_1 \to M_2$ be an isomorphism. Let $0 \neq N \ll M_2$. Then $0 \neq \varphi^{-1}(N) \ll M_1$ .Put $K = \varphi^{-1}(N)$. Let $f: M_1 \to K$ be a monomorphism and let $g = \varphi\big|_K$ then $g: K \to M_2$ is a monomorphism and $g(k) = \varphi(\varphi^{-1}(N)) = N,$ hence $g: K \to N$ is a monomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$ is a monomorphism. Therefore M_2 is small compressible.

Remark (2.1.8)

A homomorphic image of a small compressible module need not be small compressible in general.

For example, Z as a Z-module is small compressible and $Z/4Z \simeq Z_4$is not small compressible.

Proposition (2.1.9)

Let $M = M_1 \oplus M_2$ be an R-module such that $annM_1 + annM_2 = R$. Then M is small compressible if and only if M_1 and M_2 are small compressible.

Proof:

($\Rightarrow$) Follows from propositon (2.1.6).

(⇐) Let $0 \neq N \ll M$. Then by [31,proposition 4.2,p.28], $N = K_1 \oplus K_2$ for some $0 \neq K_1 \leq M_1 \leq M$ and $0 \neq K_2 \leq M_2 \leq M$. And as $N \ll M$, then $K_1 \ll M_1$ and $K_2 \ll M_2$ by [22,proposition 1.1.4,p.11]. But M_1 and M_2 are small compressible, so there are monomorphisms $f: M_1 \to K_1$ and $g: M_2 \to K_2$. Define $h: M \to N$ by $h(a, b) = (f(a), g(b))$. It can be easily checked that h is a monomorphism and hence *M* is small compressible.

Corollary (2.1.10)

Let $\{M_i\}_{i=1}^{n}$ be a finite family of small compressible *R*-modules such that $\sum_{i=1}^{n} annM_i = R$. Then $\oplus_{i=1}^{n} M_i$ is also small compressible.

"**Definition (2.1.11)[28]**

An *R*-module *M* is called ***small prime*** if $annM = annN$ for each non-zero small submodule *N* of *M*".

"**Definition (2.1.12)[28]**

A proper submodule *N* of an *R*-module *M* is called ***small prime submodule*** if and only if whenever $r \in R$ and $x \in M$ with $(x) \ll M$ and $rx \in N$ implies either $x \in N$ or $r \in [N:M]$".

Every prime module is small prime but not conversely, for example, Z_6 as a *Z*-module is small prime but not prime, while Z_4 is not small prime *Z*-module.

"**Definition (2.1.13)[21]**

An *R*-module *M* is called ***small uniform*** if every non-zero small submodule of *M* is essential in *M*".

Every uniform is small uniform but the converse need not be true in general, for instance the *Z*-module Z_6 is small uniform but not uniform.

The following results where given in [21].

"**Proposition (2.1.14) [21,lemma 2.3.3,p.56]**

If *M* is small compressible, then *M* is S-prime and *M* is S-uniform"

"**Proposition (2.1.15)[21,theorem2.3.6,p.57]**

Let $M = Rm_1 \oplus Rm_2 \oplus \ldots \oplus Rm_k$, where $m_1, m_2 \ldots, m_k \in M$.If M is small uniform and small prime, then *M* is small compressible.

We introduce in the following theorem some characterizations of small compressible modules.

Theorem (2.1.16)

Let *M* be an *R*-module. Then the following statements are equivalent:

(1) *M* is small compressible.

(2) *M* is isomorphic to an *R*-module of the form *A/P* for some small prime ideal *P* of *R* and an ideal *A* of *R* containing *P* properly.

(3) *M* is isomorphic to a non-zero submodule of a finitely generated small uniform, small prime *R*-module.

Proof:

(1)⟹(2)

Let $0 \neq m \in M$ and $Rm \ll M$. Then *Rm* is small compressible by proposition (2.1.5). Therefore *Rm* is small prime submodule of *M* by proposition (2.1.14). By (1), there is a monomorphism, say $f: M \to Rm$ and hence *M* is isomorphic to a submodule of *Rm*. On the other hand, $Rm \simeq R/ann(m)$.Moreover *M* is small compressible gives M is small prime by proposition (2.1.14),and according to [28,proposition 3.11,p.5], $ann(m)$ is a prime ideal and hence small prime ideal of R. Put $ann(m) = p$. Then $M \simeq A/P$ where A is an ideal of R contains P properly and P is a small prime ideal of R.

(2)⟹(3)

By (2), $M \simeq A/P$ for some small prime ideal P of R and an ideal A of R containing P properly, so A/P is a non-zero submodule of R/P.By hypothesis and according to [28,Examples and Remarkes 2.2,(2)], P is a prime ideal of R, R/P is an integral domain and hence R/P is a small prime R-module [28,Examples and Remarks 3.2, (6)], and R/P is a finitely generated uniform and hence small uniform R-module, hence (3) follows.

(3) ⟹(1)

By (3), M is isomorphic to a non-zero submodule of a finitely generated small uniform and small prime R-module, say $\grave{M}$, so $\grave{M}$ is small compressible R-module by proposition (2.1.15). Hence M is also small compressible R-module by proposition (2.1.7) which proves (1).

In the following theorem we give a necessary condition for a quotient module to be small compressible.

Theorem (2.1.17)

Let N be a proper submodule of an R-module M. If M/N is small compressible, then N is small prime submodule of M.

Proof:

Let $r \in R, x \in M, (x) \ll M$ and $rx \in N$. Suppose that $x \notin N$. Then $N \subsetneq N + (x)$. We claim that $\frac{N+(x)}{N} \ll \frac{M}{N}$. Suppose that $\frac{N+(x)}{N} + \frac{L}{N} = \frac{M}{N}$, for some submodule L of M containing N. Hence $\frac{N+(x)+L}{N} = \frac{M}{N}$ so $\frac{(x)+L}{N} = \frac{M}{N}$

implies that $(x)+L=M$. But $(x)\ll M$ by hypothesis, therefore $L=M$ and $\frac{L}{N}=\frac{M}{N}$ which means that $\frac{N+(x)}{N}\ll\frac{M}{N}$ .Therefore there exists a monomorphism, say $f\colon\frac{M}{N}\to\frac{N+(x)}{N}$ (since $\frac{M}{N}$ is small compressible by hypothesis).Now, we prove that $rf\left(\frac{M}{N}\right)=N$. Let $y\in f\left(\frac{M}{N}\right)$.Then $y=f(m+N)$ for some $m\in M$ on the other hand $f(m+N)\in\frac{N+(x)}{N}$, So, $y=(n+tx)+N=tx+N$ for some $n\in N$ and $t\in R$.But $ry\in rf\left(\frac{M}{N}\right)$ and $ry=rf(m+N)=r(tx+N)=t(rx)+N=N$ (since $rx\in N$).Thus $rf(\frac{M}{N})\subseteq N$ and hence $rf\left(\frac{M}{N}\right)=N=f(r.\frac{M}{N})$, then $r\frac{M}{N}=N=\frac{rM+N}{N}$ and $rM+N=N$ that is $rM\subseteq N$. Hence $r\in[N\colon M]$ which proves that *N* is small prime.

<u>Theorem (2.1.18)</u>

Let *M* be an *R*-module in which every cyclic submodule of *M* is small in *M*. Let *N* be a small prime submodule of *M* such that $[N\colon M]\not\supseteq[K\colon M]$ for each submodule *K* of *M* containing *N* properly. Then *M/N* is small compressible.

Proof:

Assume that *N* is a small prime submodule of *M*. We have to show that *M/N* is small compressible. Let $0\neq L/N\ll M/N$. Then $[N\colon M]\not\supseteq[L\colon M]$ (by hypothesis) and hence there exists $t\in[L\colon M]$ and $t\notin[N\colon M]$. Define $f\colon M/N\to L/N$ by $f(m+N)=tm+N$ for all $m\in M$. Clearly, f is a homomorphism. To prove *f* is a monomorphism. Suppose that $f(m_1+N)=f(m_2+N)$ with $m_1,m_2\in M$. Then $tm_1-tm_2=t(m_1-m_2)\in N$. But by hypothesis $(m_1-m_2)\ll M$ and *N* is small prime submodule of *M*, moreover $t\notin[N\colon M]$, therefore $m_1-m_2\in N$ and hence $m_1+N=m_2+N$. Hence *f* is a monomorphism which completes the proof.

The following are some consequences of theorem (2.1.17) and (2.1.18)

Corollary (2.1.19)

Let M be a small prime R-module such that $annM \not\supseteq [K{:}M]$ for all non-zero submodule K of M and every cyclic submodule of M is small in M. Then M is small compressible.

Proof:

Since M is small prime, then (0) is a small prime submodule of M [28] and since $annM = [0{:}M] \not\supseteq [K{:}M]$ by hypothesis therefore by the theorem (2.1.18) we get M is small compressible.

Corollary (2.1.20)

Let M be an R-module such that $annM \not\supseteq [K{:}M]$ for each submodule K of M and every cyclic submodule of M is small in M. Then M is small prime if and only if M is small compressible.

Corollary (2.1.21)

Let M be a multiplication R-module, N be a proper submodule of M and every cyclic submodule of M is small in M. Then M/N is small compressible is and only if N is small prime submodule of M.

Proof:

As M is a multiplication module, then $[N{:}M] \not\supseteq [K{:}M]$for all submodule K of M containing N properly. So according to theorem (2.1.17) and (2.1.18) the result follows.

Corollary (2.1.22)

Let I be a proper ideal of a ring R such that every principal ideal of R is small in R. Then R/I is small compressible if and only if I is a small prime ideal of R.

"Proposition (2.1.23) [21,proposition (2.3.7),p 59]

Let M be a faithful finitely generated multiplication R-module, then M is small compressible if and only if for each $(0) \neq I \ll R, ann_M I = (0)$".

"**Proposition (2.1.24)[21,proposition 2.3.9,p.60]**

Let M be a faithful finitely generated multiplication R-module. Then M is small compressible module if and only if R is small compressible ring".

"**Definition (2.1.25)[٢1]**

A small compressible module M is called ***small critically compressible*** if M cannot be embedded in any proper quotient module M/N with $0 \neq N \ll M$".

Proposition (2.1.26)

A non-zero small submodule of a small critically compressible module is also small critically compressible.

Proof:

Let M be a small critically compressible module and $0 \neq N \ll M$. Then by proposition (2.1.5) N is small compressible. Let $0 \neq H \ll N$. Then $H \ll M$ and $N/H \ll M/H$ [22,proposition 1.1.2,p.10]. Suppose that there exists a monomorphism say $\alpha: N \to N/H$. But M is small compressible implies that there is a monomorphism say $f: M \to N$. Then the composition $M \xrightarrow{f} N$

$\xrightarrow{\alpha} N/H \xrightarrow{i} M/H$ gives a monomorphism from M into M/H which is a contradiction. Therefore N is small critically compressible.

Proposition (2.1.27)

A direct summand of a small critically compressible is small critically compressible.

Proof:

Let $M = A \oplus B$ be a small critically compressible module. Then M is small compressible and by proposition (2.1.6), A is also small compressible. Let $0 \neq K \ll A$. Then $K \simeq K \oplus 0 \ll M$.Let $f: M \to K$ be a monomorphism, and suppose that there is a monomorphism say, $g: A \to A / K$. Then the composition $M \xrightarrow{f} K \xrightarrow{i} A \xrightarrow{g} A / K \xrightarrow{j} M / K$ is a monomorphism (where i and j are the inclusion homomorphisms). Therefore a contradiction. Hence A is small critically compressible.

We introduce the following concept:

Definition (2.1.28)

A small partial endomorphism of a module M is a homomorphism from a small submodule of M into M.

Examples (2.1.29)

(1) If $0 \neq N \ll M$ (M is any R-module), then the inclusion homomorphism $i: N \to M$ is a small partial endomorphism of M.

(2) Let $N = (\bar{2})$ be the submodule of the Z-module Z_8 . Then $N \ll M$ and

$f: (\bar{2}) \to Z_8$ defined by $f(\bar{x}) = 2\bar{x}$ for all $\bar{x} \in N$. Then f is a small partial endomorphism.

"**Definition (2.1.30)[31]**

Let M be an R-module. A submodule N of M is said to be ***stable***, if $f(N) \subseteq N$ for each R-homomorphism $f: N \to M$."

"M is called ***fully stable*** in case each submodule of M is stable."

Proposition (2.1.31)

Let M be a fully stable module. If M is small critically compressible module. Then every non-zero small partial endomorphism of M is a monomorphism.

Proof:

Let $0 \neq N \ll M$ and $f: N \to M$ be a non-zero small partial endomorphism. Then $f(N) \leq N$ (since M is fully stable) and $N \ll M$ gives $f(N) \ll M$ [18,Lemma 5.1.3,p.108] on the other hand $N/kerf \simeq f(N)$, So there exists an isomorphism say $\varphi: N/kerf \to f(N)$.But M is small critically compressible (by hypothesis) implies that there exists a monomorphism, say $g: M \to f(N)$, so the composition $M \xrightarrow{g} f(N) \xrightarrow{\varphi^{-1}} N/kerf \xrightarrow{i} M/kerf$ is a monomorphism and $kerf \leq N \ll M$ gives $kerf \ll M$. Thus M is embedded in $M/kerf$ which is a contradiction then, $kerf = 0$, so f is a monomorphism.

The following proposition is a partial converse of proposition (2.1.31)

Proposition (2.1.32)

Let M be a small compressible module such that the quotient of every submodule of M by a small submodule is small. If every small partial endomorphism of M is a monomorphism, then M is small critically compressible.

Proof:

Suppose that M is not small critically compressible then there is a non-zero small submodule N of M and a monomorphism $f: M \to M/N$. Therefore M is isomorphic to a submodule, say K/N of M/N with K is a submodule of M containing N. By hypothesis $K/N \ll M/N$ and since $N \ll M$ implies $K \ll M$ [٢2, proposition 1.1.2, p.10]. Hence the composition $K \xrightarrow{\pi} K/N \xrightarrow{\varphi^{-1}} M$ (where $\varphi: M \to K/N$ is an isomorphism) is a monomorphism (by hypothesis) and hence $0 = \ker(\varphi^{-1}\pi) = ker\pi = N$ which is a contradiction, therefore M is small critically compressible.

2.2 Small Retractable Modules

In this section we study the concept of small retractable modules in some details.

"**<u>Definition (2.2.1)[21]</u>**

An R-module M is called ***small retractable*** if $Hom_R(M,N) \neq 0$ for each non-zero small submodule N of M".

A ring R is called ***small retractable*** if the R-module R is small retractable. That is $Hom_R(R,I) \neq 0$ for each non-zero small ideal I of R.

<u>Examples and Remarks (2.2.2)</u>

(1) Every retractable module is small retractable and the converse is not always hold. Consider the following example:

In example (1.2.6), we show that $I = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad a,b \in R\right\}$ is not retractable, on the other hand the only small submodule of I is the zero submodule, hence I is small retractable.

(2) If M is a hollow module, then M is retractable if and only if M is small retractable.

(3) The Z-module Q is not small retractable since $Z \ll Q$ but $Hom_R(Q,Z)$= 0.

(4) Every integral domain is a small retractable ring but not conversely, for

instance Z_6 as a Z_6-module is small retractable but $Z_6$is not an integral domain.

(5) Every semisimple module is small retractable, however the converse is not true in general, for example Z is small retractable Z-module but it is not semisimple.

(6) Every module over a semisimple ring is small retractable.

(7) Every small compressible module is small retractable and the converse is not true in general, for example the Z-module, Z_{24} is small retractable but not small compressible since $\{\bar{0}, 1\bar{2}\}$ is the only small submodule in Z_{24} and $f: Z_{24} \to \{\bar{0}, \overline{12}\}$ such that $f(\bar{x}) = 12\bar{x}$ for all $\bar{x} \in Z_{24}$ is a homomorphism which is not monomorphism.

(8) Let M be an R-module. Then M is a small retractable R-module if and only if M is a small retractable $R/annM$-module.

Proposition (2.2.3)

Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is small retractable, then every non-zero small submodule of M is also small retractable.

Proof:

As in the proof of proposition (1.2.7).

Proposition (2.2.4)

Let M_1 and M_2 be two isomorphic R-modules. Then M_1 is small retractable if and only if M_2 is small retractable.

Proof:

As in the proof of proposition (1.2.3).

Remark (2.2.5)

A direct summand (and a homomorphic image, or a quotient module) of a small retractable module may not be small retractable in general.

For example, the Z-module $Z \oplus Z_{p^\infty}$ is small retractable, however Z_{p^∞} is not small retractable, $M/Z \simeq Z_{p^\infty}$ is not small retractable and Z_{p^∞} is a hollow Z-module.

In the following proposition we investigate the direct sum of small retractable modules.

Proposition (2.2.6)

If M_1 and M_2 are small retractable modules such that $annM_1$ +ann$M_2 = R$ then $M_1 \oplus M_2$ is also small retractable.

Proof:

Let $0 \neq K \ll M_1 \oplus M_2$. As $annM_1 + annM_2 = R$ by [31,proposition 4.2,p.28] gives $K = N_1 \oplus N_2$ with $N_1 \leq M_1$ and $N_2 \leq M_2$.But $N_1 \oplus N_2 \ll M_1 \oplus M_2$ implies $N_1 \ll M_1$ and $N_2 \ll M_2$ [22,proposition 1.1.4,p.11].Therefore $Hom(M_1, N_1) \neq 0$ and $Hom(M_2, N_2) \neq 0$. Let $0 \neq f: M_1 \to N_1$ and $0 \neq g: M_2, \to N_2$.Define $h: M_1 \oplus M_2 \to N_1 \oplus N_2$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ clearly h is a homomorphism. If $h = 0$, then $h(m_1, m_2) = 0$ for all $m_1 \in$

$M_1, m_2 \in M_2$,so $f(m_1) = 0$ and $g(m_2) = 0$ for all $m_1 \in M_1, m_2 \in M_2$, which is a contradiction since $f \neq 0$ and $g \neq 0$. Therefore $Hom(M_1 \oplus M_2, K) \neq 0$.

In the following proposition we give a sufficient condition for a small retractable module to be retractable.

Proposition (2.2.7)

Let M be a small retractable module. If every non-zero submodule of M contains a non-zero small submodule then M is retractable.

Proof:

Let $0 \neq N \leq M$. By hypothesis N contains a non-zero small submodule. Let $0 \neq K \ll N$. Then $K \ll M$ [22,proposition 1.1.3,p.11]. Hence $Hom(M, K) \neq 0$ (since M is small retractable), and therefore $Hom(M, N) \neq 0$ so M is retractable.

As it was mentioned in Examples and Remarks(2.2.2,7) that every small compressible module is small retractable and the converse need not be true in general, we recall in the following results that the converse holds under certain conditions:

Proposition (2.2.8)

If M is a small retractable quasi-Dedekind R-module , then every non-zero element of $Hom(M, N)$ is a monomorphism for any non-zero small submodule N of M.

Proof:

Let $0 \neq N \ll M$ and let $f: M \to N$ be a non-zero homomorphism. Then $if \in End(M)$ and $if \neq 0$. For if $if = 0$, then $if(M) = f(M) = 0$ implies $f = 0$

which is a contradiction. Hence $0 \neq if \in End(M)$ and by hypothesis if is a monomorphism which gives that f is a monomorphism.

"**Corollary (2.2.9)[21,proposition 2.3.21,p.65]**

Let M be a small retractable module. If M is quasi-Dedekind, then M is small compressible".

Corollary (2.2.10)

Let M be a finitely generated quasi-Dedekind R-module,then M is small retractable if and only if M is small prime and small uniform.

Proof:

From corollary (2.2.9), M is small compressible and according to proposition (2.1.14) and (2.1.15), the result follows.

"**Definition (2.2.11)**

A module M is called ***monoform*** if for each non-zero submodule N of M, every non-zero $f \in Hom(N, M)$ is a monomorphism" [25]. And "M is called ***S-monoform*** if for each non-zero small submodule N of M every non-zero $f \in Hom(N, M)$ is a monomorphism"[21].

Corollary (2.2.12)

Let M be a small retractable quasi-Dedekind module. Then M is S-monoform if and only if each non-zero small submodule of M is quasi-Dedekind.

Proof:

By corollary (2.2.9), M is small compressible and by [٢1,corollary 2.3.20,p.65], M is S-monoform.

2.3 Some Characterizations of Small Retractable Modules

We shall introduce some characterizations of small retractable modules

Proposition (2.3.1)

An R-module M is called small retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $Im\, f \subseteq N$ for each non-zero small submodule N of M.

Proof:

(⇒) Suppose that M is small retractable. Let $0 \neq N \ll M$. Then $Hom_R(M, N) \neq 0$. Let $g: M \to N$ be a non-zero homomorphism and $f = ig$ where $i: N \to M$ be the inclusion homomorphism, then $f \in End_R(M)$ and $f \neq 0$ since $g \neq 0$ and i is a monomorphism. Clearly, $f(N) = g(N) \subseteq N$.

(⇐) Let $0 \neq N \ll M$. By hypothesis, there exists a non-zero endomorphism $f: M \to M$ and $f(M) \subseteq N$. Therefore $f: M \to N$ is a non-zero homomorphism this completes the proof.

The following is another characterization of small retractable modules

Proposition (2.3.2)

An R-module M is small retractable if and only if for each $0 \neq x \in M$ with $x \ll M$, $Hom_R(M, Rx) \neq 0$.

Proof:

(⇒) Is obvious.

(⇐) To prove M is small retractable. Let $0 \neq N \ll M$ and let $0 \neq x \in N$, then $Rx \ll N$, so by hypothesis, $Hom(M, Rx) \neq 0$ which implies that $Hom(M, N) \neq 0$ and therefore M is small retractable.

Proposition (2.3.3)

Let M be a fully invariant R-module such that $f(M)$ is a direct summand of M for each $f \in End_R(M)$. Then M is small retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $f(M)$ is small retractable.

Proof:

(⇒) Let i_M be the identity endomorphism of M then $i_M(M) = M$ is small retractable.

(⇐) To prove M is small retractable. Let $0 \neq N \ll M$. By hypothesis there is a non-zero endomorphism $f: M \to M$ and $f(M)$ is small retractable. Since $N \ll M$, then $f(N) \ll M$ [22,proposition 1.1.3,p.11], but $f(N) \leq f(M) \leq M$ and $f(M)$ is a direct summand of M (by hypothesis) implies that $f(N) \ll f(M)$ [22,corollary 1.1.5,p.12] . As $f(M)$ is small retractable, so there is a non-zero homomorphism $g: f(M) \to f(N)$. But $f(N) \subseteq N$ since N is invariant therefore the composition $M \xrightarrow{f} f(M) \xrightarrow{g} f(N) \xrightarrow{i} N$ gives $igf \in Hom(M, N)$ and $igf \neq 0$, for if $igf = 0$, then $0 = igf(M) = gf(M)$ implies $g = 0$ which is a contradiction. Therefore M is small retractable.

"**Definition (2.3.4)[9]**

An R-module M is called ***small projective*** if for each small epimorphism $f: A \to B$(where A and B are any two R-modules) and for any homomorphism $g: M \to B$ there exists a homomorphism $h: M \to A$ such that $fh = g$. That is the following diagram is commutative.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\
A \arrow[r, "f"'] & B \arrow[r] & 0 \text{ (with } kerf \ll A)
\end{tikzcd}$$

where an epimorphism $f: A \to B$ is called small epimorphism provided that $kerf \ll A$" [18].

"**Definition (2.3.5)[10]**

A ring R is called ***V-ring*** if every simple R-module is injective".

Remark (2.3.6)

If R is a commutative ring, then R is V-ring if and only if R is a Von-Neumann regular ring" [44,corollary 3.73,p.97].

In the following proposition we show that over a V-ring the class of small projective modules is contained in the class of small retractable modules.

Proposition (2.3.7)

If R is a V-ring (or a von-Neumann regular ring), then every small projective R-module is small retractable.

Proof:

Let M be a small projective R-module. Let $0 \neq x \in M$ such that $Rx \ll M$. We have to show that $Hom(M, Rx) \neq 0$. Let A be a maximal submodule of Rx. Then Rx/A is a simple R-module and hence Rx/A is injective R-module (since R is a V-ring).

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{R}x \arrow[r, "i"] \arrow[d, "\pi"'] & \mathbf{M} \arrow[dl, dotted, "f"] \\ & \mathbf{R}x\,/\mathbf{A} & \end{tikzcd}$$

Since *Rx/A* is injective implies that there exists $f: M \to Rx/A$ such that $fi = \pi$. Note that $kerf = A \leq Rx \ll M$, so $kerf \ll M$ and M being small projective implies that there exists a homomorphism $h: M \to Rx$ which makes the following diagram commutative

$$\begin{tikzcd} & \mathbf{M} \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\ \mathbf{R}x \arrow[r, "\pi"'] & \mathbf{R}x\,/\mathbf{A} \arrow[r] & \mathbf{0} \end{tikzcd}$$

That is $\pi h = f$. We get $h \in Hom(M, Rx)$.It is left to show that $h \neq 0$. If $h = 0$, then $h(M) = 0$ and $A = \pi(0) = f(M)$. On the other hand $fi = \pi$ gives $fi(Rx) = \pi(Rx) = Rx + A$. Thus $f(Rx) = Rx + A \subseteq A$. Therefore $Rx \subseteq A$ implies $A = Rx$ which is a contradiction since A is a maximal submodule of *Rx*, and hence $h \neq 0$ which proves that *M* is small retractable.

"**<u>Definition (2.3.8)[9]</u>**

A ring R is called ***cosemisimple*** if $Rad(M) = 0$, for each R-module M. where $Rad(M) =$ the sum of all small submodules of *M*".

Proposition (2.3.9)[6,proposition 2.1.4,p.23]

A ring R is cosemisimple if and only if every R-module is small projective".

The following result follows directly from propositions (2.3.7) and (2.3.9)

Corollary (2.3.10)

If R is a cosemisimple V-ring, then every R-module is small retractable.

A relation between small uniform module and small retractable module is discussed under, certain conditions in the following proposition:

Proposition (2.3.11)

Let R be an integral domain. Then every faithful finitely generated small uniform R-module is small retractable

Proof:

Let M be a finitely generated small uniform R-module. Then $M = Rx_1 + Rx_2 + \cdots + Rx_n$ where $x_i \in M$ $\forall i = 1,2, \dots\dots, n$. Let $0 \neq N \ll M$. Then $N \leq_e$ M and hence for each $i = 1,2, \dots\dots, n$ there exists $t_i \in R$, $t_i \neq 0$ and $0 \neq t_i \ x_i \in N$ [27]. Let $t = t_1 t_2 \dots\dots. t_n$. Then $t \neq 0$ and $0 \neq$ $tx_i \in N$ for all $i = 1,2, \dots\dots, n$ and for each $m \in M$, $m = \sum_{i=1}^{n} r_i x_i$ with $r_i \in R$ $\forall i = 1,2, \dots\dots, n$.

and $tm = \sum_{i=1}^{n} t(r_i x_i) = \sum_{i=1}^{n} r_i (t x_i)$ and hence $tm \in N$, $\forall m \in M$. So we can define $f: M \to N$ by $f(m) = tm$ $\forall m \in M$. Clearly f is anon-zero homomorphism, hence $Hom(M, N) \neq 0$, for if $f = 0$, then $tm = 0$ for all $m \in M$ implies $t = 0$ (since M is faithful), but $t \neq 0$ therefore a contradiction. Hence M is retractable.

A sufficient condition for a faithful finitely generated multiplication R-module to be small retractable is that R is a small retractable ring, as it is shown in the following proposition

Proposition (2.3.12)

Let M be a faithful finitely generated multiplication R-module. Then M is small retractable.

Proof:

Let $0 \neq N \ll M$. Then $N = IM$ for some non-zero ideal I of R (since M is multiplication R-module). But $N \ll M$ and M is a faithful finitely generated multiplication R-module implies that $I \ll R$ [22, proposition1.1.8, p.14] therefore $Hom(R, I) \neq 0$ since R is small retractable by (Examples and Remarks (1.2.2,(1)). Let $f: R \rightarrow I$ be a non-zero homomorphism. Put $f(1) = a$ for some $a \in I$ and $a \neq 0$. Define $g: M \rightarrow N$ by $g(m) = am$ for all $m \in M$. It can be easily checked that g is a well-defined homomorphism, if $g = 0$, then $am = 0$ for all $m \in M$ and therefore $a \in ann(M)$, hence $a = 0$ (since M is faithful) but $a \neq 0$, therefore a contradiction and hence $Hom(M, N) \neq 0$. Therefore M is small retractable.

Remark (2.3.13)

The ring Z is small retractable but the Z-module Q is not small retractable, in fact Q is not finitely generated multiplication Z-module. This means that these two conditions cannot be dropped in the proposition (2.3.12).

Corollary (2.3.14)

Every faithful cyclic R-module is also small retractable.

Proof:

By (1.3.23), every faithful cyclic R-module is retractable and hence is small retractable.

The concept epi-retractable module was given in [5] as follows:

"**Definition (2.3.15)**

A module M is called ***epi-retractable*** if every submodule of M is a homomorphism image of M".

Example: Every semisimple module is epi-retractable but not conversely, Z as a Z-module is epi-retractable but not semisimple.

It is clear that an epi-retractable module is retractable

Now, we give the following proposition:

Proposition (2.3.16)

If M is an epi-retractable module, then every non-zero submodule of M is also an epi-retractable.

Proof:

Let $0 \neq N \leq M$ and $0 \neq K \leq N$. As M is an epi-retractable, then there exists epimorphisms $f: M \to N$ and $g: M \to K$. Define $h: N \to K$ such that $h(f(x)) = g(x)$ for all $x \in N$, h is well-defined, for if $x_1 = x_2$ in N, then $g(x_1) = g(x_2)$, thus $h(f(x_1)) = h(f(x_2))$.h is a homomorphism since g is a homomorphism and $h \neq 0$ since $g \neq 0$. Therefore N is retractable, Moreover h is an epimorphism since g is an epimorphism.

Now, we present the concept of small epi-retractable module as in the following definition:

Definition (2.3.17)

A module M is called ***small epi-retractable*** if every small submodule of M is a homomorphic image of M. That is, whenever N is a small submodule of M, then there exists an epimorphism from M onto N.

Examples and Remarkes (2.3.18)

(1) Every small epi-retractable module is small retractable.

(2) Z_4 as a Z-module is a small epi-retractable. Since $(\bar{2}) \ll Z_4 . f: Z_4 \rightarrow (\bar{2})$ is such that $f(x) = 2x$ is an epimorphism.

(3) Z as a Z-module is a small epi-retractable since Z has no non-zero small submodules.

(4) Every semisimple R-module is a small epi-retractable and not conversely by (1).

(5) Z_{p^∞} as a Z-module is not small epi-retractable.

(6) If M is a hollow module then M is small epi-retractable if and only if M is epi-retractable.

Proposition (2.3.19)

A non-zero small submodule of small epi-retractable module is also small epi-retractable.

Proof:

Let M be a small epi-retractable module and $0 \neq N \ll M$. Let $0 \neq K \ll N$. Then $K \ll M$[18,Lemma 5.1.3,p.108] Therefore there are epimorphisms

$f: M \to N$ and $g: M \to K$. Define $h: N = f(M) \to K = g(M)$ by $h(f(m)) = g(m)$ for all $m \in M$. Clearly $h \in Hom(N, K)$ and $h \neq 0$, for if $h = 0$. Then $h(f(M)) = 0 = g(M) = K$ which is a contradiction. Moreover h is an epimorphism, since $h(N) = h(f(M)) = g(M) = K$. Thus N is small epi-retractable.

Corollary (2.3.20)

A direct summand of small epi-retractable is also small epi-retractable.

Proposition (2.3.21)

Let *M* be a small epi-retractable module and *N* be a small submodule of *M*. Then *M/N* is small epi-retractable.

Proof:

Let $\bar{0} \neq K/N \ll M/N$, where K is a proper submodule of M containing *N* properly. Since N is small in M and K/N is small in M/N implies that K is small in M [22,proposition (1.1.2),p.10]. Hence there is an epimorphism, say $f: M \to K$ (since M is small epi-retractable by hypothesis). f induces a homomorphism $\bar{f}: M/N \to K/N$ with $\bar{f}(m + N) = f(m) + N$ for all $m \in M$.$\bar{f} \neq 0$,for if $\bar{f} = 0$, then $\bar{0} = \bar{f}(M/N)$= $f(M) + N = K + N$ (since f is an epimorphism). Hence $K + N = N$ implies $K = N$ which is a contradiction. Therefore $Hom(M/N, K/N) \neq 0$. Moreover $\bar{f}(M/N) = K/N$. Thus *M/N* is small epi-retractable.

Proposition (2.3.22)

Let $M_1$and M_2be two small epi-retractable modules such that $annM_1 + annM_2 = R$. Then $M_1 \oplus M_2$ is also small epi-retractable.

Proof:

Let $0 \neq N \ll M_1 \oplus M_2$. Then by [31,proposition 4.2,p.28],$N = N_1 \oplus N_2$ for some $0 \neq N_1 \leq M_1$ and $0 \neq N_2 \leq M_2$. And as $N \ll M$, then $N_1 \ll M_1$ and $N_2 \ll M_2$ by [22,proposition 1.1.4,p.11].Therefore there are epimorphisms $f: M_1 \to N_1$ and $g: M_2 \to N_2$.Define $h: M_1 \oplus M_2 \to N$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ for all $(m_1, m_2) \in M_1 \oplus M_2$. Clearly, h is a non-zero homomorphism and h is an epimorphism. Therefore $M_1 \oplus M_2$ is small epi-retractable.

<u>Corollary (2.3.23)</u>

Let $\{M_i\}_{i=1}^{n}$be a finite family of small epi-retractable modules such that $\sum_{i=1}^{n} annM_i = R$. Then $\oplus_{i=1}^{n} M_i$ is also small epi-retractable.

Chapter Three

Purely Compressible Modules and Purely Retractable Modules

Introduction

We present in this chapter another generalization of compressible modules and retractable modules namely, purely compressible modules and purely retractable modules. A detailed study is given about these concepts. The chapter includes four sections. Section one is devoted for purely compressible modules, where we present the definition with many examples and remarks, moreover many interesting properties of such modules are investigated. In the second section, we introduce and study a special type of purely compressible modules, namely purely critically compressible modules. In section three, we present the concept of purely retractable modules with many examples and properties of such modules. Some characterizations of purely retractable modules are given in the last section of this chapter; also we give the concept of purely epi-retractable module with some of its basic properties.

3.1 Purely Compressible Modules

We present in this section the concept of purely compressible module and study its basic properties; also the relation between this concept and certain types of modules is studied.

In the beginning we need to recall some concepts and some results which are related to the subject of this section.

"**Definition (3.1.1)[14]**

A submodule N of an R-module M is called ***pure*** if $N \cap IM = IN$ for each ideal *I* of *R*".

Examples (3.1.2)

(1) (0) and *M* are pure submodules of any module *M*.

(2) "Every non-zero cyclic submodule of the Z-module Q is not pure" [٣2,Example 1.2.6,p.17].

(3) $\{\bar{0}, \bar{2}\}$ is not a pure submodule of the *Z*-module Z_4.

(4) Each of $\{\bar{0}, \bar{3}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\}$ is a pure submodule of the *Z*-module Z_6.

"**Definition (3.1.3)[16]**

An *R*-module *M* is called ***pure simple*** if (0) and *M* are only pure submodules of *M*".

Every simple module is purely simple, but not conversely, for example, Z_4 is purely simple but not simple module.

Example (3.1.4)

(1) Each of *Z* and Z_4 as a *Z*-module is pure simple.

(2) Every integral domain is pure simple but not conversely.

We recall some properties of pure submodules in the following remark:

"Remark (3.1.5)[٣2,Remarks 1.2.8,p.19]

Let N and K be submodules of an R-module M. Then

(1) If N is a direct summand of M, then N is pure in M.

(2) If N is pure in M and K is pure in N, then K is pure in M.

(3) If N is pure in M and $K \leq N$, then N/K is pure in M/K.

(4) If $K \leq N \leq M$ such that K is pure in M and N/K is pure in M/K, then N is pure in M.

(5) If $K \cap N$ is pure in K, then N is pure in $N + K$.

(6) If $N + K$ is pure in M and $N \cap K$ is pure in K, then N is pure in M."

"Definition (3.1.6)[23]

A ring R is called ***regular ring*** (in the sense of Von-Neumann) if for every $r \in R$ there exists $t \in R$ such that $r = rtr$".

"Definition (3.1.7)[٤7]

An R-module M is called ***regular module*** if for every $m \in M$ and for all $r \in R$, there exists $t \in R$ such that $rm = rtrm$".

"Proposition (3.1.8)[47]

(1) Every module over a regular ring is regular.

(2) An R-module M is regular if and only if every submodule of M is pure".

Now, we introduce a new generalization of compressible modules, namely purely compressible module as in the following definition:

Definition (3.1.9)

An R-module M is called ***purely compressible*** if M can be embedded in each of its non-zero pure submodule. That is M is purely compressible if there exists a monomorphism $f: M \to N$ whenever N is a non-zero pure submodule of M.

A ring R is called ***purely compressible*** if R as an R-module is purely compressible.

Examples and Remarks (3.1.10)

(1) Every compressible module is purely compressible, however there are purely compressible modules which are not compressible.

For example, Z_4 as a Z-module is purely compressible, but Z_4 is not compressible.

(2) Every purely simple module is purely compressible and the converse need not be true in general.

(3) If R is an integral domain, then R is a purely compressible R-module but not conversely.

(4) If M is a regular module, then M is compressible if and only if M is purely compressible.

(5) If R is a regular ring and M is an R-module, then M is compressible if and only if M is purely compressible.

Now, we need to recall and prove the following lemma:

Proposition (3.1.11)

Let M_1 and M_2 be two isomorphic modules. Then M_1 is purely compressible if and only if M_2 is purely compressible.

Proof:

Suppose that $M_1$is purely compressible and $\varphi: M_1 \to M_2$be an isomorphism. Let N be a non-zero pure submodule of M_2. Let $K = \varphi^{-1}(N)$, then K is a submodule of M_1. We claim that K is pure in M_1. Let I be an ideal of R.

But f is a monomorphism gives $\varphi(IM_1 \cap K) = \varphi(IM_1) \cap \varphi(K) = I\varphi(M_1) \cap \varphi\varphi^{-1}(N) = \mathrm{I}M_2 \cap N = IN = I\varphi(K) = \varphi(IK)$. But φ is an isomorphism, then $IM_1 \cap K = IK$. Hence K is pure in M_1. Let $f: M_1 \to K$ be a monomorphism and let $g = \varphi|_K$ then $g: K \to M_2$ is a monomorphism and $g(K) = \varphi(\varphi^{-1}(N)) = N,$ hence $g: K \to N$ is a monomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$,is a monomorphism . Therefore M_2 is purely compressible.

Proposition (3.1.12)

A non-zero pure submodule of a purely compressible module is purely compressible.

Proof:

Let M be a purely compressible module and N be a non-zero pure submodule of M. Let K be a pure submodule of N. Then K is pure in M (by Remark (3.1.5),(2)). Therefore there is a monomorphism say $f: M \to K$ and hence $if: K \to N$ is also a monomorphism where $i: K \to N$ is the inclusion homomorphism. Thus N is purely compressible.

Corollary (3.1.13)

Every direct summand of a purely compressible module is a purely compressible.

Corollary (3.1.14)

Let M be a regular module. If M is purely compressible, then every non-zero submodule of M is purely compressible.

Corollary (3.1.15)

Let R be a regular ring and M be a purely compressible R-module. Then every non-zero submodule of M is purely compressible.

Remark (3.1.16)

A homomorphic image (or a quotient) of purely compressible module need not be purely compressible in general. For example, Z as a Z-module is purely compressible but $Z/6Z \simeq Z_6$ is not a purely compressible Z-module.

Also this example shows the conditions R is a regular ring in corollary (3.1.15), cannot be discarded.

Remark (3.1.17)

The direct sum of purely compressible modules is not necessarily purely compressible. Consider the following example.

Example (3.1.18)

Let $M = Z_4 \oplus Z_2$ as a Z-module. Each of Z_4 and Z_2 is purely compressible Z-module. But M is not purely compressible as it is shown below:

$M = \{(\bar{0},\bar{0}),(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{0}),(\bar{2},\bar{1}),(\bar{3},\bar{0}),(\bar{3},\bar{1})\}$,

$A = Z(\bar{1},\bar{1}) = \{(\bar{0},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{0}),(\bar{3},\bar{1})\}$, and $B = Z(\bar{2},\bar{1}) = \{(\bar{0},\bar{0}),(\bar{2},\bar{1})\}$.

Clearly, $M = A \oplus B$ and hence each of A and B is a pure submodule of M, but each of A and B does not contain a copy of M, that is M cannot be embedded in A (or in B). Therefore M is not purely compressible.

Now, we introduce the following concepts:

"**Definition (3.1.19)[43]**

An R-module M is called ***purely prime*** if $ann(M) = ann(N)$ for each non-zero pure submodule N of M. "

Clearly, every prime module is purely prime; but not conversely. For instance, the Z-module Z_4 is purely prime but not prime. While Z_6 as a Z-module is not purely prime (in fact it is not prime).

Definition (3.1.2٠)

A submodule N of a module M is called ***purely prime submodule*** if whenever $rx \in N$ with $r \in R, x \in M$ and (x) is pure in M implies either $x \in N$ or $r \in [N:M]$.

Example (3.1.2١)

Let $M = Z_6$ as a Z-module and $N = (\bar{2})$. N is pure in Z_6 and N is purely prime submodule of M.

Lemma (3.1.2٢)

An R-module M is purely prime if and only if (0) is a purely prime submodule of M.

Proof:

($\Longrightarrow$) Suppose that $rx = 0$ with $r \in R, x \in M$ and (x) is pure in M. Assume that $x \neq 0$. Since M is purely prime (by hypothesis) implies that $annM = ann(x)$ and hence $r \in annM = [0:M]$ hence (0) is a purely prime submodule of M.

($\Longleftarrow$) Suppose that (0) is a purely prime submodule of M, let N be a non-zero pure submodule of M and let $r \in annN$. Then $rx = 0$ for all $x \in N$, and hence $rx \in (0)$. Assume that $x \neq 0$, then $r \in [0:M] = annM$, therefore $annN \subseteq annM$, so $annM = annN$, thus M is purely prime.

<u>Lemma (3.1.23)</u>

Let M be a module in which every submodule of a pure submodule is also pure. If M is purely prime module, then $annN$ is a prime ideal of R for each non-zero pure submodule N of M.

Proof:

Let N be a non-zero pure submodule of M. let $a, b \in R$ and $ab \in annN$. Then $abN = 0$. Suppose that $bN \neq 0$. But $bN \leq N$ and N is pure in M. By hypothesis bN is pure in M, but M is purely prime and $a \in annbN$ implies $a \in annM$,on the other hand $annM = annN$,so $a \in annN$ and hence $annN$ is a prime ideal of R.

The converse of Lemma (3.1.23) is not true in general

For example: Z_6 is not purely prime Z-module, however $ann_z(\bar{2}) = 3Z$ and $ann_z(\bar{3}) = 2Z$ which are both prime ideals in Z and that $(\bar{2}), (\bar{3})$ are pure submodule of Z_6.

Proposition (3.1.24)

Every purely compressible module is purely prime.

Proof:

Let M be a purely compressible module. Let N be a non-zero pure submodule of M. We have to show that $annM = annN$. Let $r \in annN$. Then $rN = 0$. Let $f: M \to N$ be a monomorphism, then $f(rM) = rf(M) \subseteq rN = 0$ implies that $rM = 0$, thus $r \in annM$ and therefore $annM = annN$.

"**Definition (3.1.25)[4]**

A module M is said to have the ***pure sum property*** (PSP) if the sum of any two pure submodules of M is pure in M".

Proposition (3.1.26)

Let M be a module having PSP and N be a proper submodule of an R-module M. If M/N is purely compressible, then N is purely prime submodule of M.

Proof:

Let $rx \in N$ with $r \in R, x \in M$ and (x) is pure in M. suppose that $x \notin N$. We have to show that $r \in [N: M]$. $x \notin N$ implies $N < N + (x)$. Since M has PSP by hypothesis, then $N + (x)$ is a pure submodule of M and hence $\frac{N+(x)}{N}$ is a pure submodule of $\frac{M}{N}$ by (Remark 3.1.5,(3)). But $\frac{M}{N}$ is purely compressible (by hypothesis), therefore there exists a monomorphism, say $f: \frac{M}{N} \to \frac{N+(x)}{N}$. We can prove $rf\left(\frac{M}{N}\right) = N$ as in the proof of (2.1.17). But f is a monomorphism, implies that $\frac{rM}{N} = N$, thus $rM \subseteq N$ and therefore $r \in [N: M]$ hence N is purely prime submodule of M.

We note that the converse of proposition (3.1.26) hold in case every cyclic submodule of M is pure in M as we shall show in the following result.

Proposition (3.1.27)

Let M be a module such that every cyclic submodule of M is pure in M. If N is a proper purely prime submodule of M such that $[N: M] \not\supseteq [K: M]$ for all submodules K of M containing N properly. Then M/N is purely compressible.

Proof:

Let L/N be a pure submodule of M/N with L is a submodule of M containing N properly. By hypothesis $[N: M] \not\supseteq [L: M]$, so there exists $t \in [L: M]$ and $t \notin [N: M]$. Define $f: M/N \to L/N$ such that $f(m+N) = tm + N$ for all $m \in M$. Clearly f is a homomorphism. To prove f is a monomorphism. Let $m + N \in kerf$.Then $f(m+N) = N$, so $tm + N = N$ implies $tm \in N$. As N is purely prime submodule of M and (m) is pure in M, moreover $t \notin [N: M]$, therefore $m \in N$(by definition (3.1.20), so $kerf = N$ and hence f is a monomorphism, whence M/N is purely compressible.

In order to give some applications of proposition (3.1.26), the following lemmas are needed

"Lemma (3.1.28)[33,proposition 2.4.5,p.58]

Every multiplication module has PSP".

"Lemma (3.1.29)[33,theorem 2.4.6,p.58]

A ring R is regular if and only if every R-module has PSP".

Corollary (3.1.30)

Let *M* be a multiplication module and *N* be a proper pure submodule of *M* If *M/N* is purely compressible, then *N* is purely prime submodule of *M*.

Proof:

M being a multiplication module implies that $[N:M] \neq [K:M]$ for any two distinct submodules *N* and *K* of *M*, moreover *M* has PSP by lemma (3.1.28). Hence the result follows by proposition (3.1.26).

Corollary (3.1.31)

Let *M* be a cyclic module and *N* is a proper pure submodule of *M*. If *M/N* is purely compressible, then *N* is purely prime submodule.

Proof:

As *M* is a cyclic module gives *M* is a multiplication module, and according to corollary (3.1.30), we get the result.

Corollary (3.1.32)

Let *R* be a regular ring and *N* is a proper submodule of *M* . If *M/N* is purely compressible, then *N* is purely prime submodule.

Proof:

By proposition (3.1.8), *M* is a regular module and *N* is a pure submodule of *M*. And by lemma (3.1.29), *M* has PSP. Hence *N* is purely prime submodule by proposition (3.1.26).

Corollary (3.1.33)

Let M be an R-module such that $annM \not\supseteq [K: M]$ for each non-zero submodule K of M. Then M is purely compressible if and only if (0) is a purely prime submodule of M, if and only if M is a purely prime module.

Next we present the concept of purely uniform module.

Definition (3.1.34)

An R-module M is called ***purely uniform*** if the intersection of any two non-zero pure submodules of M is non-zero.

Equivalently, M is purely uniform if every non-zero pure submodule of M is essential in M.

Equivalently, M is purely uniform if every non-zero pure submodule of M is purely essential in M.

Clearly every uniform module is purely uniform.

Remark (3.1.35)

A non-zero pure submodule N of a module M is purely essential if and only if for each $0 \neq x \in M$ with Rx is a pure submodule of M there exists $0 \neq r \in R$ such that $0 \neq rx \in N$.

Proof:

($\Longrightarrow$) Is clear

($\Leftarrow$) Let K be a non-zero pure submodule of M. Let $0 \neq x \in K$ with Rx is a pure submodule of M. Then $0 \neq rx \in N$ for some $0 \neq r \in R$ (by hypothesis). Therefore $0 \neq rx \in N \cap K$ implies N is purely essential in M.

Proposition (3.1.36)

Every purely compressible module is purely uniform.

Proof:

Let M be a purely compressible module. Let $0 \neq x \in M$ such that Rx is a pure submodule in M and let $f: M \to Rx$ be a monomorphism. Then $f(x) = rx$ for some $0 \neq r \in R$. Let $0 \neq m \in M$ and let $f(m) = tx \neq 0$ for some $0 \neq t \in R$. Then $f(rm) = rf(m) = r(tx) = t(rx) = tf(x) = f(tx)$ and hence $rm = tx \in Rx$ and $rm \neq 0$. For if $rm = 0$, then $0 = f(rm) = tx = f(m)$ gives $m = 0$ which is a contradiction. So Rx is purely essential in M and hence M is purely uniform.

In the class of faithful finitely generated multiplication modules we give the following characterization of purely compressible modules:

Theorem (3.1.37)

Let M be a faithful finitely generated multiplication R-module. Then M is purely compressible if and only if for each non-zero pure ideal I of R, $ann_M(I) = 0$.

Proof:

($\Longrightarrow$) Let I be a non-zero pure ideal of R. Then $N = IM$ is a pure submodule of M [30, theorem 1.4,p.67] but M is purely compressible implies M is purely prime (by proposition (3.1.24), and hence $ann_R(M) = ann_R(N) = ann_R(IM) = ann_R I$.Therefore $ann_R(I) = 0$(since M is faithful). Now, to prove $ann_M(I) = 0$. Let $ann_M(I) = KM$ for some ideal K of R we have $I ann_M(I) = 0$ and hence $IKM = 0$ implies $IK \subseteq ann_R M = 0$, so $IK = 0$, therefore $K \subseteq ann_R(I) = 0$, so $K = 0$ and hence $ann_M(I) = 0$

($\Longleftarrow$) To prove M is purely compressible. Let N be a non-zero pure submodule of M. then $N = IM$ for some non-zero pure ideal I of R [30,theorem 1.4,p.67]. Let $0 \neq a \in I$ and define $f: M \to N$ by $f(m) = am$ for all $m \in M$. Clearly f is a well-defined homomorphism. Let $m \in kerf$. Then $am = 0$ therefore $m \in ann_M(a)$. but $(a) \leq I$ and I is pure in R implies (a) is pure in R (since M is faithful multiplication module) by [30,p.65]. Hence $ann_M(a) = 0$ (by hypothesis), so $m = 0$ and therefore $kerf = 0$ which gives M is purely compressible.

Corollary (3.1.38)

Let M be a faithful finitely generated multiplication R-module. Then M is purely compressible if and only if $Hom_R(R/I,M) = 0$ for each non-zero pure ideal I of R.

Proof:

By [31,lemma 2.7,p.45], $ann_M(I) \simeq Hom_R(R/I,M)$ for each ideal I of R hence the result follows according to theorem (3.1.37).

Since every cyclic module is a multiplication module, the following are also consequences of theorem (3.1.37).

Corollary (3.1.39)

Let M be a faithful cyclic R-module. Then M is purely compressible if and only if $ann_M(I) = 0$ for each non-zero pure ideal I of R.

Corollary (3.1.40)

A ring R is purely compressible if and only if $ann_R(I) = 0$ for each non-zero pure ideal I of R.

Proposition (3.1.41)

Let M be a faithful finitely generated multiplication R-module. Then M is purely compressible if and only if R is purely compressible ring.

Proof:

($\Longrightarrow$) Let I be a non-zero pure ideal of R. we have to show that $ann_R(I) = 0$. Let $r \in ann_R(I)$. Then $rI = 0$ and hence $rIM = 0$ implies that $rM \subseteq ann_M(I)$. But M is purely compressible by hypothesis and according to theorem (3.1.37), $ann_M(I) = 0$, hence $rM = 0$, so $r \in ann_R(M) = 0$ since M is faithful and hence $r = 0$. Therefore $ann_R(I) = 0$ and by corollary (3.1.40), R is purely compressible ring.

($\Longleftarrow$) Let I be a non-zero pure ideal of R. Then $N = IM$ is a non-zero pure submodule of M.[30,theorem 1.4,p.67].But R is purely compressible gives $ann_R(I) = 0$ (by corollary(3.1.40), and it can be checked easily that $ann_M(I) = \big(ann_R(I)\big)M$,therefore $ann_M(I) = 0$, so by theorem (3.1.37) , M is purely compressible.

Corollary (3.1.42)

Let M be a faithful finitely generated multiplication R-module. If M is purely prime, then M is purely compressible.

Proof:

Let I be a non-zero pure ideal of R. Then IM is a pure submodule of M [30,theorem 1.4,p.67].but M is purely prime (by hypothesis), therefore $ann_R(M) = ann_R(IM)$ by definition (3.1.19) and since M is faithful (by hypothesis)implies $ann_R(IM) = 0 = ann_R(I)$. But $ann_M(I) = (ann_R I)M = 0.\ M = 0$. Therefore $ann_M(I) = 0$ implies that M is purely compressible (by theorem (3.1.37).

Corollary (3.1.43)

Let M be a faithful finitely generated multiplication R-module then M is purely compressible if and only if M is purely prime.

Proof:

Follows from proposition (3.1.24) and corollary (3.1.42).

Corollary (3.1.44)

Let M be a faithful cyclic R-module then M is purely compressible if and only if M is purely prime.

Corollary (3.1.45)

A ring R is purely compressible if and only if R is purely prime.

Before we state and prove the next result we need to recall and prove the following two lemmas.

Lemma (3.1.46)

If M is an R-module such that every cyclic submodule is pure, then every purely prime submodule of M is prime in M.

Proof:

Let N be a purely prime submodule and $rx \in N$ with $r \in R$ and $x \in M$.By hypothesis Rx is a pure submodule of M and N is purely prime, implies that either $x \in N$ or $r \in [N: M]$ and hence N is a prime submodule of M.

Lemma (3.1.47)

Let R be a ring in which every principal ideal is pure. If P is purely prime ideal of R, then R/P is purely uniform R-module.

Proof:

Let *A/P* be a non-zero pure submodule of *R/P*. To prove *A/P* is purely essential in *R/P*. Let $x + P \neq P$ in *R/P* with $R(x + P)$ is pure in *R/P*. Let $a + P \neq P$ in *A/P*. Note that $x \notin P$ and $a \notin P$ we claim that $ax \notin P$. Suppose that $ax \in P$.As P is purely prime and *Rx* is pure in *R* gives either $x \in P$ or $a \in [P: R]$. But $x \notin P$, so, $a \in [P: R]$ implies $aR \subseteq P$ and hence $a \in P$ which is a contradiction. Therefore $P \neq ax + P = a(x + P) \in A/P$ which implies that *A/P* is purely essential and hence *R/P* is purely uniform.

Theorem (3.1.48)

Let *R* be a ring in which every principle ideal is pure. Let *M* be a faithful finitely generated multiplication *R*-module such that every submodule of a pure submodule is also pure. Then the following statements are equivalent:

(1) *M* is purely compressible.

(2) *M* is isomorphic to an *R*-module of the form *A/P* for some purely prime ideal *P* of *R* and an ideal *A* of *R* containing *P* properly.

(3) *M* is isomorphic to a non-zero submodule of a finitely generated purely uniform, purely prime *R*-module.

Proof:

(1)⟹(2)

Let $0 \neq m \in M$ and Rm is pure in *M*. Then Rm is purely compressible by proposition (3.1.12). Hence Rm is purely prime by proposition (3.1.24). As *M* is purely compressible, then there exists a monomorphism, say $f: M \to Rm$ that is *M* is isomorphic to a submodule of Rm. But $Rm \simeq R/ann(m)$ and *M* is purely prime *R*-module implies that *ann(m)* is prime, and hence purely prime ideal of R by lemma (3.1.23). Let $P = ann(m)$. Then $Rm \simeq R/P$ and M is isomorphic to a submodule of R/P, say A/P where A is an ideal of R containing P properly and P is a purely prime ideal of R.

(2) ⟹(3)

By (2), $M \simeq A/P$ for some purely prime ideal P of R and an ideal A of R containing P properly, hence A/P is a non-zero submodule of the finitely generated module R/P. But P is purely prime ideal of R implies that R/P is purely prime R-module by lemma(3.1.22) and by Lemma (3.1.47) A/P is purely uniform R-module.

(3) ⟹ (1)

By (3), M is isomorphic to a non-zero submodule of a finitely generated purely uniform and purely prime R-module, say $\grave{M}$, and according to (corollary (3.1.42) $\grave{M}$ is purely compressible R-module hence M is purely compressible R-module (by proposition (3.1.1١)).

<u>Corollary (3.1.49)</u>

Let R be a ring in which every principle ideal is pure. Let M be a cyclic faithful R-module such that every submodule of a pure submodule is also pure. Then the following statements are equivalent:

(1) M is purely compressible.

(2) M is isomorphic to an R-module of the form A/P for some purely prime ideal P of R and an ideal A of R containing P properly.

(3) M is isomorphic to a non-zero submodule of a finitely generated purely uniform, purely prime R-module.

3.2. Purely Critically Compressible Modules

A special type of purely compressible modules is given and studied in this section, namely purely critically compressible module.

Definition (3.2.1)

An R-module M is called ***purely critically compressible*** if M is purely compressible and M cannot be embedded in any of its quotient modules M/N with N is a non-zero proper pure submodule of M.

Examples and Remarks (3.2.2)

(1) Every critically compressible module is purely critically compressible. In particular Z as a Z-module is purely critically compressible, in fact it is critically compressible.

(2) Z_n as a Z-module is not purely critically compressible $\forall n > 1$.

(3) The Z-module Q is not purely critically compressible.

(4) Z_{p^∞} as a Z-module is not purely critically compressible.

(5) If R is a regular ring (in the sense of von Neumann), then R as an R-module is critically compressible if and only if R is purely critically compressible. This is follows from the fact that R is a regular ring if and only if every ideal of R is pure.

(6) If M is a regular module then M is purely critically compressible if and only if M is critically compressible.

Proposition (3.2.3)

Let M be a purely critically compressible module then every non-zero pure submodule of M is also purely critically compressible.

Proof:

Let *N* be a non-zero pure submodule of *M*. Then *N* is purely compressible (by proposition (3.1.12). Let H be a pure submodule of *N*. Then H is pure in *M* and *N/H* is pure in *M/H* (by Remark (3.1.5, (2) and (3)). Suppose that there is a monomorphism, say $\alpha: N \to N/H$. But M is purely compressible implies that there is a monomorphism, say $f: M \to N$. Then the composition $M \xrightarrow{f} N \xrightarrow{\alpha} N/H \xrightarrow{i} M/H$ is a monomorphism where *i* is the inclusion homomorphism. So *M* is embedded in *M/H* which is a contradiction since *M* is purely critically compressible. Hence *N* is purely critically compressible.

Corollary (3.2.4)

A non-zero direct summand of a purely critically compressible is also purely critically compressible.

Now, we need to introduce the following concept:

Definition (3.2.5)

A pure partial endomorphism of a module *M* is a homomorphism from a pure submodule of *M* into *M*.

Examples (3.2.6)

(1) If *N* is a pure submodule of a module *M*, then the inclusion homomorphism $i: N \to M$ is a pure partial endomorphism of *M*.

(2) If *N* is a direct summand of an *R*-module *M*, then every homomorphism from *N* into *M* is a pure partial endomorphism of *M*.

(3) If *M* is a regular module (or a semisimple module). Then every partial endomorphism of *M* is a pure partial endomorphism of *M*.

Proposition (3.2.7)

Let M be a fully stable module in which every submodule of a proper pure submodule is also pure. If M is purely critically compressible, then every non-zero pure partial endomorphism of M is a monomorphism.

Proof:

Let N be a non-zero pure submodule of M and $f: N \to M$ be a non-zero partial endomorphism then $N/kerf \simeq f(N)$. By hypothesis each of $kerf$ and $f(N)$ is pure in M, and since M is purely compressible then there is a monomorphism, say $g: M \to f(N)$. Then the composition

$M \xrightarrow{g} f(N) \xrightarrow[\text{iso.}]{\varphi} N/kerf \xrightarrow[\text{incl.}]{i} M/kerf$ is an embedding of M into $M/kerf$ which is a contradiction since M is purely critically compressible. Therefore $kerf = 0$ and hence f is a monomorphism.

Proposition (3.2.8)

Let M be a purely compressible module such that the quotient of every submodule of M by a pure submodule is pure. If every non-zero pure partial endomorphism of M is a monomorphism, then M is purely critically compressible.

Proof:

Assume that M is not purely critically compressible. Therefore there is a non-zero pure submodule N of M and a monomorphism $f: M \to M/N$. Hence M is isomorphic to a submodule say K/N of M/N. By hypothesis K/N is pure in M/N and since N is pure in M implies K is pure in M (by Remark (3.1.5),(4)). The composition $K \xrightarrow{\pi} K/N \xrightarrow[\text{iso.}]{\varphi} M$ is a pure partial endomorphism of M.

So by hypothesis $\varphi\pi$ is a monomorphism and hence $\ker(\varphi\pi) = 0 = ker\pi = N$ which is a contradiction.

Proposition (3.2.9)

Every purely critically compressible module is indecomposable but not conversely.

Proof:

Let M be a purely critically compressible module. Suppose that M is decomposable. Then $M = A \oplus B$ with A and B are non-zero proper pure submodules of M. So, $B \simeq M/A$. Let $\alpha: B \to M/A$ be an isomorphism but M is purely compressible, hence there is a monomorphism say $f: M \to B$ and therefor $\alpha f: M \to M/A$ is a monomorphism, which is a contradiction.

For the converse Q as a Z-module is indecomposable but not purely critically compressible.

3.3 Purely Retractable Modules

In this section we introduce and study the concept purely retractable module as a generalization of retractable module. Some characterizations of such modules are given. Moreover, the relationships between this concept and some other types of module are also investigated.

Definition (3.3.1)

An R-module M is called ***purely retractable*** if $Hom(M, N) \neq 0$ for each non-zero pure submodule N of M.

A ring R is called ***purely retractable*** if the R-module R is purely retractable, that is $Hom_R(R, I) \neq 0$ for each non-zero pure ideal I of R. "where an ideal I

of R is called pure if $IJ = I \cap J$ for each ideal J of R" [47,proposition 1.3,p.8].

Examples and Remarks (3.3.2)

(1) Every retractable module is purely retractable, but the converse is not true in general. Consider example (1.2.6) where $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \in R\}$, S is a retractable ring and $I = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\}$ is a non-retractable S-module. We claim that I is purely retractable. Let $J = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R\}$ and $K = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in R\}$, J and K are the only non-zero proper submodules of I and it is clear that $I = J \oplus K$ and hence J and K are pure in I. Also it is clear that $Hom_s(I, J) \neq 0$ and $Hom_s(I, K) \neq 0$ so, I is a purely retractable S-module.

(2) Every pure simple module is purely retractable. The Z-modules Z, Z_4 and Z_{P^∞} are pure simple. On the other hand Z_{p^∞} is not retractable Z-module.

(3) Every purely compressible module is purely retractable and the converse need not be true in general. For example, Z_6 as a Z-module is purely retractable but not purely compressible.

(4) If M is a regular module, then M is purely retractable if and only if M is retractable.

(5) If R is a regular ring and M is an R-module, then M is purely retractable if and only if M is retractable.

(6) Every semisimple (simple) module is purely retractable.

(7) Let M be an R-module. Then M is purely retractable R-module if and only if M is purely retractable $R/annM$-module.

Proposition (3.3.3)

If M_1 and M_2 are two isomorphic modules, then M_1 is purely retractable if and only if M_2 is so.

Proof:

As in the proof (3.1.11) and proof (1.2.3).

Proposition (3.3.4)

Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is purely retractable, then every non-zero pure submodule of M is also purely retractable.

Proof:

As in the proof of proposition (1.2.7).

Corollary (3.3.5)

Let M be a module such that $End(M)$ is a Boolean ring. If M is purely retractable, then every direct summand of M is also purely retractable.

Proposition (3.3.6)

If N is a proper purely prime submodule of a module M such that $[N:M] \not\supseteq [K:M]$ for all submodules K of M containing N properly, then M/N is purely retractable.

Proof:

Let L/N be a pure submodule of $M/N$with L is a submodule of M containing N properly. By hypothesis $[N:M] \not\supseteq [L:M]$, so there exists $t \in [L:M]$ and $t \notin [N:M]$. Define $f: M/N \to L/N$ such that $f(m+N) = tm + N$ for all $m \in M$. Clearly f is a homomorphism and $f \neq 0$, if $f = 0$ then $tm + N = N$, $tm \in$

N for all $m \in M$. Thus $t \in [N: M]$ which is a contradiction. Therefore M/N is purely retractable.

We are now going to investigate when a purely retractable is purely compressible.

Proposition (3.3.7)

Let M be a purely retractable quasi-Dedekind R-module, then M is a purely compressible.

Proof:

Let N be a non-zero pure submodule of M and let $f: M \to N$ be a non-zero homomorphism. then $fi: M \to M$ be an endomorphism on M, where $i: N \to M$ is the inclusion homomorphism. By hypothesis if is a monomorphism and hence f is a monomorphism. Therefore M is purely compressible.

Corollary (3.3.8)

Let M be a purely retractable quasi-Dedekind module. Then M is purely prime and purely uniform.

Proof:

By corollary (3.3.7), M is purely compressible and according to proposition (3.1.24) and (3.1.36), M is purely prime and purely uniform.

Corollary (3.3.9)

If M is a purely retractable quasi-Dedekind module in which the quotient of every submodule of M by a pure submodule of M is also pure, then M is purely critically compressible.

Proof:

By corollary (3.3.7), M is purely compressible and by proposition (3.2.8), M is purely critically compressible.

3.4 Some Characterizations of Purely Retractable Modules

We introduce in this section necessary and (or) sufficient conditions for a module to be purely retractable.

As a characterization of purely retractable module we have the following proposition

Proposition (3.4.1)

Let M be a module. Then M is purely retractable if and only if there exists $0 \neq \varphi \in End_R(M)$ such that $Im\,\varphi \subseteq N$ for each non-zero pure submodule N of M.

Proof:

($\Rightarrow$) Suppose that M is purely retractable. Let N be a non-zero pure submodule of M. Then $Hom_R(M,N) \neq 0$. Let $0 \neq f: M \to N$ be a non-zero homomorphism. Let $\varphi = if$ where $i: N \to M$ be the inclusion homomorphism, then $\varphi \in End_R(M)$, $\varphi \neq 0$ and $Im\varphi = if(M) = f(M) \subseteq N$.

($\Leftarrow$) To prove M is purely retractable, let N be a non-zero pure submodule of M. By hypothesis, there exists a non-zero endomorphism $\varphi: M \to M$ such that $Im\varphi = \varphi(M) \subseteq N$, hence $\varphi: M \to N$ is a non-zero homomorphism, thus $0 \neq \varphi \in Hom(M,N)$, therefore M is purely retractable.

<u>Proposition (3.4.2)</u>

Let M be a module such that every cyclic submodule of a pure submodule of M is pure in M. Then M is purely retractable if and only if $Hom(M, Rx) \neq 0$ for each $0 \neq x \in M$ with Rx is pure in M.

Proof:

(⇒) Obvious.

(⇐) Let N be a non-zero pure submodule of M. Let $0 \neq x \in M$. By hypothesis Rx is pure in M and $Hom(M, Rx) \neq 0$ and hence $Hom(M, N) \neq 0$, therefore M is purely retractable.

<u>Corollary (3.4.3)</u>

Let M be a module in which every cyclic submodule of M is pure. Then M is purely retractable if and only if $Hom(M, Rx) \neq 0$ for each $0 \neq x \in M$.

<u>Corollary (3.4.4)</u>

Let R be a regular (Von-Neumann) ring. Then every projective R-module is purely retractable (In fact retractable).

Proof:

Let M be a projective R-module and let $0 \neq x \in M$. Since R is a is regular ring, then Rx is a direct summand of M, [44,Exercies 17,p.57]. Therefore Rx is a pure submodule of M (by Remark (3.1.5),(1)) and $Hom(M, Rx) \neq 0$. Hence M is purely retractable (by corollary (3.4.3)).

In order to give other consequences of proposition (3.4.2) we need to recall the following definition

"**Definition (3.4.5)[44]**

An R-module M is called ***finitely presented*** (f.p.) if there exists a short exact sequence $0 \to K \xrightarrow{f} F \xrightarrow{g} M \to 0$ such that F is a finitely generated free R-module and K is a finitely generated R-module".

Corollary (3.4.6)

Every finitely presented module is purely retractable.

Proof:

Let M be a f.p. module. Let $0 \neq x \in M$ such that Rx is pure in M. Then Rx is a direct summand of M [44, ,Exercies 32,p.163]. Therefore $\mathrm{Hom}(M, Rx) \neq 0$ and according to corollary (3.4.3) M is purely retractable.

Corollary (3.4.7)

Every finitely generated projective module is purely retractable.

Proof:

Let M be a finitely generated projective module. Then M is finitely presented [44, ,Exercies 1,p.159] and by corollary (3.4.6) M is purely retractable.

In the following proposition we also give a sufficient condition for a module to by purely retractable.

Proposition (3.4.8)

Let M be a module such that every non-zero pure submodule of M contains a non-zero direct summand of M. Then M is purely retractable.

Proof:

Let N be a non-zero pure submodule of M. By hypothesis there is $0 \neq A \leq N$ and A is a direct summand of M. So, $M = A \oplus B$ for some $0 \neq B < M$. Let $\rho_A : M \to A$ be the projection homomorphism. Therefore $\rho_A \in Hom(M, A)$ and $i\rho_A \in Hom(M, N)$ where $i: A \to N$ is the inclusion homomorphism.

If $i\rho_A = 0$, then $0 = i\rho_A(M) = \rho_A(M) \simeq A$, implies $A = 0$ which is a contradiction therefore $Hom(M, N) \neq 0$, hence M is purely retractable.

While the following proposition gives a sufficient condition for a purely retractable module to be retractable:

<u>Proposition (3.4.9)</u>

Let M be a module such that every non-zero submodule of M contains a non-zero direct summand of M. if M is purely retractable, then M is retractable.

Proof:

Let $0 \neq N \leq M$. By hypothesis there is a direct summand of M, say K and $K \subseteq N$. Then K is pure in M (Remark (3.1.5),(1)) As M is purely retractable implies $Hom(M, K) \neq 0$ and hence $Hom(M, N) \neq 0$. Therefore M is retractable.

<u>Corollary (3.4.10)</u>

Let M be a module such that every non-zero submodule of M contains a non-zero direct summand of M. Then M is retractable if and only if M is purely retractable.

Now, the following definition is needed:

"**Definition (3.4.11)[32]**

A module M is called ***purely lifting module*** if for every submodule N of M, there exists a pure submodule K of M such that $K \subseteq N$ and $N/K \ll M/K$".

The following corollary is a direct consequence of proposition (3.4.9):

Corollary (3.4.12)

If M is a purely lifting module, then M is retractable if and only if M is purely retractable.

"**Definition (3.4.13)[34]**

An R-module M is called a ***V-module***, if for every factor module N of M,$Rad(N) = 0$".

Proposition (3.4.14)

Let M be a V-module. If M is purely lifting, then M is retractable if and only if M is purely retractable.

Proof:

As M is a V-module, then M is purely lifting if and only if M is a regular module [32,proposition 2.2.4,p.40]. And according to (Examples and Remarks (3.3.2),(4)), M is retractable if and only if M is purely retractable.

Proposition (3.4.15)

Let M be a finitely generated multiplication R-module,then M is purely retractable module.

Proof:

Let N be a non-zero pure submodule of M. Then $N = IM$ for some non-zero ideal I of R. We claim that I is pure in R. Let J be an ideal of R. Then $JM \cap N = JM \cap IM = (J \cap I)M$ (since M is faithful multiplication), but IM is pure in M. gives $JM \cap IM = J(IM) = (JI)M$. Hence $(J \cap I)M = (JI)M$, so $J \cap I = JI$[48,proposition 3.4,p.55]. Therefore I is a pure ideal in R. But R is purely retractableby (Examples and Remarks 1.2.2,(1))implies that $Hom(R, I) \neq 0$. Let $0 \neq f: R \to I$ be a homomorphism. Let $f(1) = a$. Then $a \neq 0$. Define $g: M \to N$ by $g(m) = am$ for all $m \in M$ clearly, g is a well-defined homomorphism, and $g \neq 0$ since M is faithful. Therefore $Hom(M, N) \neq 0$ which is what we wanted.

<u>Corollary (3.4.16)</u>

Every faithful cyclic R-module is also purely retractable.

Now, we present the concept of purely epi-retractable module as in the following definition:

<u>Definition (3.4.17)</u>

A module M is called ***purely epi-retractable*** if every pure submodule of M is a homomorphic image of M. That is, whenever N is a pure submodule of M, then there exists an epimorphism from M onto N.

<u>Examples and Remarkes (3.4.18)</u>

(1) Every purely epi-retractable module is purely retractable

(2)$I = \{\begin{pmatrix} a & b \\ 0 & o \end{pmatrix} : a, b \in R\}$ in (3.3.2,(1)) is purely epi-retractable S-module.

(3) Q as a Z-module is not epi-retractable and hence not purely epi-retractable.

(4) Every semisimple module is purely epi-retractable.

(5) Every pure simple module is purely epi-retractable.

(6) If M is a regular module (or R is a regular ring), then M is purely epi-retractable if and only if M is epi-retractable.

Proposition (3.4.19)

A non-zero pure submodule of purely epi-retractable module is also purely epi-retractable.

Proof:

Let M be a purely epi-retractable module and let N be a non-zero pure submodule of M. let K be a non-zero pure submodule of N. Then K is pure in M (by Remark 3.1.5,(2)). Therefore there are epimorphisms $f: M \to N$ and $g: M \to K$. Define $h: N = f(M) \to K = g(M)$ by $h\big(f(m)\big) = g(m)$ for all $m \in M$. Clearly $h \in Hom(N, K)$ and $h \neq 0$, for if $h = 0$. Then $h\big(f(M)\big) = 0 = g(M) = K$ which is a contradiction. Moreover h is an epimorphism, since $h(N) = h\big(f(M)\big) = g(M) = K$. Thus N is purely epi-retractable.

Corollary (3.4.20)

A direct summand of purely epi-retractable is also purely epi-retractable.

Proposition (3.4.21)

Let M be a purely epi-retractable module and N be a pure submodule of M. Then M/N is purely epi-retractable.

Proof:

Let $\bar{0} \neq K/N$ be a pure submodule of M/N, where K is a proper submodule of M containing N properly. Since N is pure in M and K/N is pure in M/N implies that K is pure in M (by Remark 3.1.5,(4)). Hence there is an epimorphism, say $f: M \to K$ (since M is purely epi-retractable by hypothesis). f induces a homomorphism $\bar{f}: M/N \to K/N$ with $\bar{f}(m+N) = f(m) + N$ for all $m \in M$.$\bar{f} \neq 0$,for if $\bar{f} = 0$, then $\bar{0} = \bar{f}(M/N) = f(M) + N = K + N$ (since f is an epimorphism). Hence $K + N = N$ implies $K = N$ which is a contradiction. Therefore $Hom(M/N, K/N) \neq 0$. Moreover $\bar{f}(M/N) = K/N$. Thus M/N is purely epi-retractable.

Proposition (3.4.22)

Let $M_1$and M_2be two purely epi-retractable modules such that $annM_1 + annM_2 = R$. Then $M_1 \oplus M_2$ is also purely epi-retractable.

Proof:

Let N be a non-zero pure submodule of $M_1 \oplus M_2$. Then by [31,proposition 4.2,p.28] $N = N_1 \oplus N_2$ for some submodule N_1 of M_1 and $N_2$of M_2. By [13,proposition 4.2] N_1 is pure in M_1 and N_2 is pure in M_2. Therefore there are epimorphisms $f: M_1 \to N_1$ and $g: M_2 \to N_2$.Define $h: M_1 \oplus M_2 \to N$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ for all $(m_1, m_2) \in M_1 \oplus M_2$. Clearly, h is a non-zero homomorphism and h is an epimorphism. Therefore $M_1 \oplus M_2$ is purely epi-retractable.

Corollary (3.4.23)

Let $\{M_i\}_{i=1}^{n}$be a finite family of purely epi-retractable modules such that $\sum_{i=1}^{n} annM_i = R$. Then $\oplus_{i=1}^{n} M_i$ is also purely epi-retractable.

Chapter Four

Primely Compressible Modules and Primely Retractable Modules

Introduction

The last generalization for compressible and retractable modules in our work is given by using the concept of prime submodules. This is the subject of this chapter, where we introduce in this chapter the concepts of primely compressible and primely retractable modules. The chapter consists of five sections. In the first section we introduce the concepts of generalized prime modules and generalized prime submodule which are basic concepts in our study of the subject of chapter four, where we give this concepts with some of their basic properties which are needed in the next sections. In section two, we give the concept of primely compressible modules with some examples, basic properties, characterizations and the relationships of such modules with some types of modules. In the third section, we give and study a sort of primely compressible modules, namely primely critically compressible modules. The forth section is devoted to primely retractable modules, where we give the definition with many examples; also we investigate the basic properties of such modules. In section five, we give necessary and (or) sufficient conditions for modules to be primely retractable. Moreover in this section, we present the concept of primely epi-retractable modules with some examples and study some of its basic properties.

4.1 Generalized Prime Modules

In this section we shall introduce the concepts generalized prime module and generalized prime submodules and study some of their properties that are related to our work in the next sections of this chapter.

"**Definition (4.1.1)[29]**

A module M is called ***fully prime*** if every proper submodule of M is a prime submodule".

Z_p as a Z-module is fully prime for each prime number P.

Examples and Remarks (4.1.2)

(1) Each of Q and Z as Z-modules is prime module.

(2) Z is not prime submodule of Q. In fact (0) is the only prime submodule of Q. while pZ is a prime submodule of Z for each prime number p.

(3) "A module M is torsion-free if and only if M is a prime and faithful module"[12,remark 1.1,p.33].

(4) "Every direct summand of a prime module is a prime submodule" [12,proposition 1.2,p.34]

(5) "A module M is prime if and only if 0 is a prime submodule of M"[29,p.303]

(6) "Let p be a prime number. Then the Z-module Z_{p^∞} has no prime submodules"[2,Example and Remark 1.1.20,p.34].

As a generalization of prime module and prime submodule we introduce the concepts generalized prime module and generalized prime submodule as follows:

Definition (4.1.3)

An R- module M is called ***a generalized prime*** module if $ann(M) = ann(N)$ for each non-zero prime submodule N of M.

Examples (4.1.4)

(1) Every prime module is generalized prime but not conversely in general, for example the Z-module Z_{p^∞} is a generalized prime, since it is primeless (has no prime submodules) but it is not prime.

(2) Every simple module is generalized prime.

(3) Every torsion-free fully prime module is a generalized prime.

Definition (4.1.5)

A submodule N of a module M is called ***generalized prime submodule*** if whenever $rx \in N$ with $r \in R$ and $x \in M$ and Rx is a prime submodule of M, then either $x \in N$ or $r \in [N: M]$.

Every prime submodule is a generalized prime submodule but not conversely.

For example: The Z-module $Z_{12}$is not a prime submodule of Z_{12}, but it is a generalized prime submodule of Z_{12}.

Proposition (4.1.6)

Let M be a generalized prime module then (0) is a generalized prime submodule of M. The converse holds if every cyclic submodule of M is a prime submodule of M.

Proof:

($\Longrightarrow$) Let $r \in R, x \in M$ and Rx is a prime submodule of M such that $rx = 0$. If $x \neq 0$, then $annM = ann(x)$ (Since M is generalized prime by hypothesis).Hence $r \in annM = [0{:}M]$. If $x = 0$, then $x \in (0)$. Therefore (0) is generalized prime submodule of *M*.

($\Longleftarrow$) Suppose that 0 is a generalized prime submodule of *M*, let *N* be a non-zero prime submodule of *M* and let $r \in annN$. Then $rx = 0$ for all $x \in N$. So, $rx \in (0)$. Assume that $x \neq 0$, then $r \in [0{:}M]$(since (0) is a generalized prime submodule of *M* by hypothesis), but $[0{:}M] = annM$, hence $r \in annM$ gives $annN \subseteq annM$ and therefore $annM = annN$, thus *M* is generalized prime.

Corollary (4.1.7)

Let *M* be a module such that every cyclic submodule of *M* is prime. Then *N* is generalized prime submodule of *M* if and only if *M/N* is a generalized prime module.

Propsition (4.1.8)

Let *M* be an *R*-module such that a non-zero cyclic submodule of a direct summand of *M* is a prime submodule of *M*. If *M* is a generalized prime, then every non-zero direct summand of *M* is a prime submodule of *M*.

Proof:

Let *K* be a non-zero direct summand of *M*. then $M = K \oplus H$ for same non-zero submodule H of M. to prove K is a prime submodule of M. Let $rx \in K$ with $r \in R$ and $x \in M$. Then $x = a + b$ with $a \in K$ and $b \in H$, suppose that $x \notin K$ then $b \neq 0$,$rx = ra + rb$ and $rb = rx - ra \in H \cap K = (0)$. Thus $rb = (0)$ and *(b)* is a non-zero submodule of *H* by hypothesis *(b)* is a prime submodule of *M*. And *(0)* is generalized prime submodule of *M* by Proposition (4.1.6) implies that $r \in [0:M] = ann\ M$ (since $b \neq 0$). So $rM = (0)$, therefore $r \in [K:M]$ which gives K is a prime submodule of M.

"**Definition (4.1.9)[49]**

An R-module M is called ***Z-regular module*** if for all $a \in R$, there exists $x \in R$ such that $a = axa$".

Proposition (4.1.10)

Let M be a Z-regular R-module. If M is generalized prime such that a non-zero cyclic submodule of a direct summand of M is a prime submodule of M ,then $annN$ is a prime ideal of R for each non-zero cyclic prime submodule N of M.

Proof:

Let $0 \neq N = (x)$ be a prime submodule of M. Let $a, b \in R$ such that $abN = 0$.Then $abx = 0$.Suppose that $bx \neq 0$. Let $K = (bx)$, then $K \leq N$. But M is regular gives K is a direct summand of M [49,proposition 2.3,p.30], and by proposition (4.1.8) K is a prime submodule of M. But $a \in annK$ implies $a \in annM = annN$ (since M is generalized prime) therefore $annN$ is a prime ideal of R.

4.2 Primely Compressible Modules

We shall give in this section the concept of primely compressible module as a generalization of compressible modules. Many basic properties of such modules are also studied. Moreover the relationships between these types of modules are given.

Definition (4.2.1)

An R-module M is called ***primely compressible*** if M can be embedded in each of its non-zero prime submodule. That is M is purely compressible if there exists a monomorphism $f: M \to N$ whenever N is a non-zero prime submodule of M.

A ring R is ***primely compressible*** if R as an R-module is primely compressible.

Examples and Remarks (4.2.2)

(1) Every compressible is primely compressible but the converse need not be true in general, for example the Z-module Q is not compressible but primely compressible since 0 is the only prime submodule of Q.

(2) If R is an integral domain and K is the field of fraction of R, then 0 is the only prime submodule of K as an R-module [29] so, K is a primely compressible R-module.

(3) Every simple module is primely compressible.

(4) If M is a fully prime module, then M is primely compressible if and only if M is compressible.

(5) Let M be a torsion-free module such that $[N: M] = 0$ for each proper submodule N of M. Then M is primely compressible if and only if M is purely compressible.

Proof:

Since M is torsion-free and $[N: M] = 0$ for each proper submodule N of M implies that N is a prime submodule of M if and only if N is a pure submodule of M [15]. Hence the result follows.

(6) Let M be a prime faithful module such that $[N:M]=0$ for each proper submodule N of M. Then M is primely compressible if and only if M is a purely compressible.

Proof:

Since M is faithful prime module then M is torsion-free [12,Remark 1.1,p.33] and by (5) M is primely compressible if and only if M is purely compressible.

(7) Let M be a prime module such that $annM=[N:M]$ for each proper submodule N of M. Then M is primely compressible if and only if M is purely compressible.

Proof:

According to the hypothesis and[12,proposition 1.3,p.34] implies that N is prime submodule of M if and only if N is a pure submodule of M. Therefore the result follows.

Proposition (4.2.3)

If M_1 and $M_2$are isomorphic R-modules, then $M_1$is primely compressible if and only if M_2 is so.

Proof:

Assume that M_1 is primely compressible and let $\varphi: M_1 \to M_2$ be an isomorphism and N be a non-zero prime submodule of M_2.Then $\varphi(M_1) \not\subset N$, for if $\varphi(M_1) \subset N$ implies $M_2 \subseteq N$ that is $M_2 = N$ which is a contradiction since N is a prime submodule of M_2. Therefore $\varphi^{-1}(N)$ is a prime submodule of M_1[37,proposition 1.2,p.1043] So, there exists a monomorphism say $f: M_1 \to K$ where $K=\varphi^{-1}(N)$ (since M_1 is primely compressible by hypothesis). Let $g=\varphi\big|_K$. Then $g: K \to M_2$ is a monomorphism and $g(K)=$

$\varphi(K) = \varphi\big(\varphi^{-1}(N)\big) = N$. So $g: K \to N$ is a monomorphism. Now, we have the following composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$ is a monomorphism from M_2 into *N* which means that M_2 is primely compressible.

Now, the following condition is needed

(*) Let *M* be a module satisfying $\forall K \leq N \leq M$ if *N* is a prime submodule of *M* and *K* is a prime submodule of *N*, then *K* is a prime submodule of *M*.

Proposition (4.2.4)

Let *M* be a module satisfying (*). If *M* is primely compressible, then every non-zero prime submodule of *M* is also primely compressible.

Proof:

Let *M* be a primely compressible module and *N* be a non-zero prime submodule of *M* and *K* be a non-zero prime submodule of *N*.As *M* has condition (*) gives *K* is a prime submodule of *M*.Therefore there is a monomorphism say $f: M \to K$ and hence $if: K \to N$ is also a monomorphism where $i: K \to N$ is the inclusion homomorphism. Thus *N* is purely compressible.

Corollary (4.2.5)

Let *M* be a fully prime module which has condition (*). If *M* is primely compressible, then every non-zero submodule of *M* is primely compressible.

Corollary (4.2.6)

Let *M* be an *F*-module (F is a field) and *M* has condition (*). If *M* is primely compressible then every non-zero submodule of *M* is primely compressible.

Proof:

Let $0 \neq N \leq M$. Then N is a prime submodule of M since F is a field and by proposition (4.2.4) N is primely compressible.

Corollary(4.2.7)

Let M be a prime module which has the condition (*). If M is primely compressible, then every non-zero direct summand of M is primely compressible.

Proof:

Let N be a non-zero direct summand of M. Then N is a prime submodule of M by (Examples and Remarks4.1.2,(4)) and by proposition (4.2.4) N is primely compressible.

Remark (4.2.8)

The direct sum of primely compressible modules is not necessary primely compressible. Consider the following example

Example (4.2.9)

Let $M = Z_2$ as a Z-module. Clearly Z_2 is primely compressible. On the other hand $Z_2 \oplus Z_2$ is prime Z-module[29,Lemma 1.1,p.305] and hence Z_2 is a prime submodule of $Z_2 \oplus Z_2$(Remarks and Examples 4.1.4,(4)) but $Z_2 \oplus Z_2$ cannot be embedded in Z_2. Therefore $Z_2 \oplus Z_2$ is not primely compressible.

Proposition (4.2.10)

If M is primely compressible, then M is generalized prime module.

Proof:

Let M be a primely compressible module. Let N be a non-zero prime submodule of M. we have to show that $annM = annN$.Let $r \in annN$. Then $rN = 0$. Let $f: M \to N$ be a monomorphism, then $f(rM) = rf(M) \subseteq rN = 0$ implies that $rM = 0$, thus $r \in annM$ and therefore $annM = annN$.

Remark (4.2.11)

The converse of proposition (4.2.10) is not true in general, for example, $Z_2 \oplus Z_2$ is a prime Z-module and hence generalized prime, but it is not primely compressible.

Next we present the concept of purely uniform module.

Definition (4.2.12)

An R-module M is called ***primely uniform*** if the intersection of any two non-zero prime submodules of M is non-zero.

Equivalently, M is primely uniform if every non-zero prime submodule of M is primely essential in M.

Clearly every uniform module is primely uniform

Proposition (4.2.13)

A non-zero prime submodule N of a module M is primely essential if and only if for each $0 \neq x \in M$ with Rx is a prime submodule of M there exists $0 \neq r \in R$ such that $0 \neq rx \in N$.

Proof:

($\Longrightarrow$) Is clear

($\Leftarrow$) Let K be a non-zero prime submodule of M. Let $0 \neq x \in K$ with Rx is a prime submodule of M. Then $0 \neq rx \in N$ for some $0 \neq r \in R$ (by hypothesis). Therefore $0 \neq rx \in N \cap K$ implies N is primely essential in M.

Proposition (4.2.14)

Every primely compressile module is primely uniform

Proof:

Let N be a prime submodule of M. let $0 \neq x \in M$ such that Rx is a prime submodule of M. Then there exists a monomorphism, say $f: M \to Rx$. Let $0 \neq m \in N$. Then $f(m) = tx$ for some $0 \neq t \in R$, and $f(x) = rx$ for some $0 \neq r \in R$, $f(rm) = rf(m) = r(tx) = t(rx) = tf(x) = f(tx)$ therefore $rm = tx \in N$ and $tx \neq 0$.So N is primely essential in M and hence M is primely uniform.

In the class of faithful finitely generated multiplication modules we give the following characterization of primely compressible modules:

Theorem (4.2.15)

Let M be a faithful finitely generated multiplication R-module. If M is primely compressible, then for each non-zero prime ideal I of R, $ann_M(I) = 0$.

Proof:

Let I be a non-zero prime ideal of R. Then $N = IM$ is a prime submodule of M [46,proposition 4.6,p.28] but M is primely compressible implies M is generalized prime (by proposition(4.2.10), and hence $ann_R(M) = ann_R(N) = ann_R(IM) = ann_R(I)$, therefore $ann_R(I) = 0$ (since M is faithful). Now, to prove $ann_M(I) = 0$. Let $ann_M(I) = KM$ for some ideal K

of R we have $Iann_M(I) = 0$ and hence $IKM = 0$ implies $IK \subseteq ann_R M = 0$, so $IK = 0$, therefore $K \subseteq ann_R(I) = 0$, so $K = 0$ and hence $ann_M(I) = 0$

The converse holds in case every non-zero principal ideal of R is prime.

Theorem (4.2.16)

Let R be a ring such that every non-zero principal ideal of R is prime. If M is a faithful finitely generated multiplication and $ann_M(I) = 0$ for each non-zero prime ideal I of R, then M is primely compressible.

Proof:

Let N be a non-zero prime submodule of M. then $N = IM$ for some non-zero prime ideal I of R [46,proposition 4.6,p.28]. Let $0 \neq a \in I$ and define $f: M \to N$ by $f(m) = am$ for all $m \in M$. Clearly f is a well-defined homomorphism. Let $m \in kerf$. Then $am = 0$ therefore $m \in ann_M(a)$, but $(a) \leq I$ and I is prime in R implies (a) is prime in R by hypothesis. Hence $ann_M(a) = 0$ (by hypothesis), so $m = 0$ and therefore $kerf = 0$ which gives M is primely compressible.

Corollary (4.2.17)

Let M be a faithful finitely generated multiplication R-module. Then M is primely compressible if and only if $Hom_R(R/I,M) = 0$ for each non-zero prime ideal I of R. where every non-zero principal ideal of R is prime.

Proof:

By [31,Lemma 2.7,p.45], $ann_M(I) \simeq Hom_R(R/I,M)$ for each ideal I of R hence the result follows according to theorem (4.2.15) and theorem (4.2.16).

Since every cyclic module is a multiplication module, the following are also consequences of theorem (4.2.15) and theorem (4.2.16).

Corollary (4.2.18)

Let M be a faithful cyclic R-module. Then M is primely compressible if and only if $ann_M(I) = 0$ for each non-zero prime ideal I of R. where every non-zero principal ideal of R is prime.

Proof:

(⇒) follows directly by proposition (4.2.15)

(⇐)Follows from proposition (4.2.17)

Corollary (4.2.19)

A ring R in which every non-zero principal ideal is prime is primely compressible if and only if $ann_R(I) = 0$ for each non-zero prime ideal I of R.

Proof:

(⇒) follows directly by proposition (4.2.15)

(⇐)Follows from proposition (4.2.17)

roposition (4.2.20)

Let M be a faithful finitely generated multiplication R-module, If M is primely compressible, then R is primely compressible.

Proof:

Let I be a non-zero prime ideal of R. we have to show that $ann_R(I) = 0$. Let $r \in ann_R(I)$. Then $rI = 0$ and hence $rIM = 0$ implies that $rM \subseteq ann_M(I)$. But M is primely compressible by hypothesis and according to theorem

(4.2.15). $ann_M(I) = 0$, hence $rM = 0$, so $r \in ann_R(M) = 0$ since M is faithful and hence $r = 0$. Therefore $ann_R(I) = 0$ and by corollary (4.2.19), R is primely compressible ring.

Proposition (4.2.21)

Let R be a primely compressible ring such that every non-zero principal ideal of R is prime. If M is a faithful finitely generated multiplication module, then M is primely compressible.

Proof:

Let I be a non-zero prime ideal of R and R is primely compressible gives $ann_R(I) = 0$ by corollary(4.2.19), and it can be checked easily that $ann_M(I) = \left(ann_R(I)\right)M$ therefore $ann_M(I) = 0$, so by theorem (4.2.16) , M is primely compressible.

The following proposition is a partial converse of proposition (4.2.10)

Proposition (4.2.22)

Let M be a faithful finitely generated multiplication module. If M is generalized prime, then M is primely compressible.

Proof:

Let I be a non-zero prime ideal of R. Then IM is a prime submodule of M [46,proposition 4.6,p.28] but M is generalized prime (by hypothesis), therefore $ann_R(M) = ann_R(IM)$ by definition (4.1.٣) and since M is faithful (by hypothesis) implies $ann_R(IM) = 0 = ann_R(I)$. But $ann_M(I) = (ann_R I)M = 0.M = 0$. Therefore $ann_M(I) = 0$ implies that M is primely compressible (by theorem (4.2.16).

Corollary(4.2.23)

Let M be a faithful finitely generated multiplication R-module then M is primely compressible if and only if M is generalized prime.

Proof:

Follows from proposition (4.2.10) and proposition (4.2.22).

Corollary (4.2.24)

Let M be a faithful cyclic R-module then M is primely compressible if and only if M is generalized prime.

Corollary (4.2.25)

A ring R is primely compressible if and only if R is generalized prime.

Now, we need to state and prove the following lemma.

Lemma (4.2.26)

Let R be a ring in which every principal ideal of R is prime. If P is a generalized prime ideal of R, then R/P is primely uniform R-module

Proof:

Let $P \neq A/P$ be a prime submodule of R/P. To prove A/P is primely essential in R/P. Let $P \neq x + P \in R/P$ with $R(x + P)$ is a prime submodule of R/P. Let $P \neq a + P \in A/P$. Suppose that $ax \in P$ we have $a \notin P, x \notin P$ and Rx is prime in R implies that $a \in [P: R]$. Hence $aR \subseteq P$ gives $a \in P$ which is a contradiction. Therefore $ax \notin P$ that is $P \neq ax + P = a(x + P) \in A/P$ which completes the proof.

Theorem (4.2.27)

Let R be a ring in which every principal ideal is prime. Let M be a Z-regular faithful finitely generated multiplication R-module which satisfy condition (*)such that a non-zero cyclic submodule of a direct summand of M is prime submodule of M. Then the following statements are equivalent:

(1) M is primely compressible.

(2) M is isomorphic to an R-module of the form A/P for some generalized prime ideal P of R and an ideal A of R containing P properly.

(3) M is isomorphic to a non-zero submodule of a finitely generated primely uniform, generalized prime R-module.

Proof:

(1)⟹(2)

Let $0 \neq m \in M$ and Rm is prime in M. Then Rm is primely compressible by proposition (4.2.4). Hence Rm is generalized prime by proposition (4.2.10). As M is primely compressible, then there is a monomorphism, say $f: M \to Rm$ that is M is isomorphic to a submodule of Rm. But $Rm \simeq R/ann(m)$ and by proposition (4.2.10) M is generalized prime R-module implies that $ann(m)$ is a prime ideal of R by proposition (4.1.10). Let $P = ann(m)$. Then $Rm \simeq R/P$ and M is isomorphic to a submodule of R/P, say A/P where A is an ideal of R containing P properly.

(2) ⟹(3)

By (2), $M \simeq A/P$ for some generalized prime ideal P of R and an ideal A of R containing P properly, hence A/P is a non-zero submodule of the finitely generated module R/P. On the other hand R/P is a generalized R-module by (corollary 4.1.7) and by lemma (4.1.26), *R/P* is primely uniform and hence (3) follows.

(3) ⟹(1)

By (3), *M* is isomorphic to a non-zero submodule of a finitely generated primely uniform and generalized prime *R*-module, say $\grave{M}$,and according to (proposition(4.2.22) $\grave{M}$ is primely compressible *R*-module hence *M* is primely compressible *R*-module (by proposition (4.2.3)).

Corollary (4.2.28)

Let *R* be a ring in which every principal ideal is prime. Let *M* be a *Z*-regular cyclic faithful *R*-module which satisfy condition (*)such that a non-zero cyclic submodule of a direct summand of *M* is prime submodule of *M*. Then the following statements are equivalent:

(1) *M* is primely compressible

(2) *M* is isomorphic to an *R*-module of the form *A/P* for some generalized prime ideal *P* of *R* and an ideal *A* of *R* containing *P* properly.

(3) *M* is isomorphic to a non-zero submodule of a finitely generated primely uniform, generalized prime *R*-module.

4.3. Primely Critically Compressible Modules

A special type of primely compressible modules is given and studied in this section, namely primely critically compressible module.

Definition (4.3.1)

An R-module M is called ***primely critically compressible*** if M is primely compressible and M cannot be embedded in any of its quotient module M/N with N is a non-zero proper prime submodule of M.

Examples and Remarks (4.3.2)

(1) Every critically compressible module is primely critically compressible the converse is not true in general, for example: the Z-module Q is primely critically compressible but not critically compressible.

(2)Z as a Z-module is primely critically compressible.

(3) Z_n as a Z-module is not primely critically compressible.

(4) Z_{p^∞} as a Z-module is not primely critically compressible.

(5) If R is an integral domain and K is the field of fraction of R, then(0) is the only prime submodule of K as an R-module [29] so, K ia a primely critically compressible R-module.

(6) Every simple module is primely critically compressible.

(7) If M is a fully prime module, then M is primely critically compressible if and only if M is critically compressible.

Proposition (4.3.3)

Let M be a primely critically compressible module satisfying (*), then every non-zero prime submodule of M is also primely critically compressible.

Proof:

Let N be a non-zero prime submodule of M. Then N is primely compressible by (proposition 4.2.4).Let H be a prime submodule of N. Then H is prime in

M (since *M* satisfying (*)) and it can easily that *N/H* is prime in *M/H* Suppose that there is a monomorphism, say $\alpha: N \to N/H$. But M is primely compressible implies that there is a monomorphism, say $f: M \to N$. Then the composition $M \xrightarrow{f} N \xrightarrow{\alpha} N/H \xrightarrow{i} M/H$ is a monomorphism where i is the inclusion homomorphism. So *M* is embedded in *M/H* which is a contradiction since *M* is primely critically compressible. Hence *N* is primely critically compressible.

Corollary (4.3.4)

A non-zero direct summand of a prime and primely critically compressible module satisfying (*) is also primely critically compressible.

Now, we need to introduce the following concept:

Definition (4.3.5)

A prime partial endomorphism of a module *M* is a homomorphism from a prime submodule of *M* into *M*.

Examples (4.3.6)

(1) If *N* is a prime submodule of a module *M*, then the inclusion homomorphism $i: N \to M$ is a prime partial endomorphism of *M*.

(2) If *N* is a direct summand of a prime *R*-module *M*, then every homomorphism from *N* into *M* is a prime partial endomorphism of *M*.

(3) If *M* is a fully prime module. Then every partial endomorphism of *M* is a prime partial endomorphism of *M*.

Proposition (4.3.7)

Let M be a fully stable module which satisfying (*). If M is primely critically compressible, then every non-zero primely partial endomorphism of M is a monomorphism.

Proof:

Let N be a non-zero prime submodule of M and $f: N \to M$ be a non-zero partial endomorphism then $\frac{N}{kerf} \simeq f(N)$. Each of $kerf$ and $f(N)$ is prime in M since M satisfying(*)and since M is primely compressible then there is a monomorphism,say $f: M \to f(N)$.Then the compositin$M \xrightarrow{f} f(N)$ $\xrightarrow[\text{iso.}]{\varphi} N/kerf \xrightarrow[\text{incl.}]{i} M/kerf$ is an embedding of M into $M/kerf$ which is a contradiction since M is primely critically compressible. Therefore $kerf = 0$ and hence f is a monomorphism.

Proposition (4.3.8)

Let M be a prime module. If M is primely critically compressible, then M is indecomposable but not conversely.

Proof:

Let M be a primely critically compressible module. Suppose that M is decomposable. Then $M = A \oplus B$ with A and B are non-zero proper prime submodules of M. So, $B \simeq M/A$. Let $\alpha: B \to M/A$ be an isomorphism but M is primely compressible, hence there is a monomorphism say $f: M \to B$ and therefor $\alpha f: M \to M/A$ is a monomorphism, which is a contradiction.

For the converse Z as a Q-module is indecomposable but not primely critically compressible.

4.4 Primely Retractable Modules

In this section we introduce and study the concept primely retractable module as a generalization of retractable module. Some characterizations of such modules are given. Moreover, the relationships between this concept and some other types of module are also investigated.

Definition (4.4.1)

An R-module M is called ***primely retractable*** if $Hom(M,N) \neq 0$ for each non-zero prime submodule N of M.

A ring R is called ***primely retractable*** if the R-module R is primely retractable

Remarks and Examples (4.4.2)

(1) Every retractable module is primely retractable, but not conversely, for example: Q as a Z-module is primely retractable since 0 is the only prime submodule of Q, and Q is not retractable since $Hom(Q,Z) = 0$.

(2) If M is a fully prime module, then M is primely retractable if and only if M is retractable.

(3) Every primely compressible module is primely retractable and the converse need not be true in general. For example,Z_{12} as a Z-module is primely retractable but not primely compressible.

(4) Every simple module is primely retractable but not conversely.

(5) Let M be an R-module. Then M is primely retractable R-module if and only if M is primely retractable $R/annM$-module.

(6) Let M be a torsion-free module such that $[N{:}M] = 0$ for each proper submodule N of M. Then M is primely retractable if and only if M is purely retractable.

Proof:

As in the proof of (Examples and Remarks 4.2.2,5) and (3).

(7) Let M be a prime faithful module such that $[N{:}M] = 0$ for each proper submodule N of M. Then M is primely retractable if and only if M is a purely retractable.

Proof:

As in the proof of (Examples and Remarks 4.2.2,6) and (3).

(8) Let M be a prime module such that $annM = [N{:}M]$ for each proper submodule N of M. Then M is primely retractable if and only if M is purely retractable.

Proof:

As in the proof of (Examples and Remarks 4.2.2,7) and (3).

Proposition (4.4.3)

If M_1 and M_2 are two isomorphic modules, then M_1 is primely retractable if and only if M_2 is so.

Proof:

As in the proof of proposition (4.2.3) and proof (1.2.3).

Proposition (4.4.4)

Let M be an R-module satisfying (*) such that $End_R(M)$ is a Boolean ring. If M is primely retractable, then every submodule of M is also primely retractable.

Proof:

As in the proof of proposition (1.2.7).

Corollary (4.4.5)

Let M be a module satisfying (*) such that $End(M)$ is a Boolean ring. If M is primely retractable, then every direct summand of M is also primely retractable.

We are now going to investigate when a primely retractable module is primely compressible.

Proposition (4.4.6)

Let M be a primely retractable quasi-Dedekind R-module, then M is a primely compressible.

Proof:

Let N be a non-zero prime submodule of M and let $f: M \rightarrow N$ be a non-zero homomorphism. Then $fi: M \rightarrow M$ is an endomorphism on M, where $i: N \rightarrow M$ is the inclusion homomorphism. By hypothesis if is a monomorphism and hence f is a monomorphism. Therefore M is primely compressible

Corollary (4.4.7)

Let M be a primely retractable quasi-Dedekind module. Then M is generalized prime and primely uniform.

Proof:

By corollary (4.4.6), M is primely compressible and according to proposition (4.2.10) and (4.2.14), M is generalized prime and primely uniform.

5. Some Characterizations of Primely Retractable Module

As a characterization of primely retractable module we have the following proposition

Proposition (4.5.1)

Let M be a module. Then M is primely retractable if and only if there exists $0 \neq \varphi \in End_R(M)$ such that $Im\,\varphi \subseteq N$ for each non-zero prime submodule N of M.

Proof:

($\Rightarrow$) Suppose that M is primely retractable. Let N be a non-zero prime submodule of M. Then $Hom_R(M, N) \neq 0$. Let $0 \neq f: M \to N$ be a non-zero homomorphism. Let $\varphi = if$ where $i: N \to M$ be the inclusion homomorphism, then $\varphi \in End_R(M)$, $\varphi \neq 0$ and $Im\varphi = if(M) = f(M) \subseteq N$.

($\Leftarrow$) To prove M is primely retractable, let N be a non-zero prime submodule of M. By hypothesis, there exists a non-zero endomorphism $\varphi: M \to M$ such that $Im\varphi = \varphi(M) \subseteq N$, hence $\varphi: M \to N$ is a non-zero homomorphism, thus $0 \neq \varphi \in Hom(M, N)$, therefore M is primely retractable.

Proposition (4.5.2)

Let M be a module such that every cyclic submodule of a prime submodule of M is prime in M. Then M is primely retractable if and only if $Hom(M, Rx) \neq 0$ for each $0 \neq x \in M$.

Proof:

(⇒) Obvious.

(⇐) Let N be a non-zero prime submodule of M. Let $0 \neq x \in N$. By hypothesis Rx is prime in M and $Hom(M, Rx) \neq 0$ and hence $Hom(M, N) \neq 0$, therefore M is primely retractable.

Corollary (4.5.3)

Let M be a module in which every cyclic submodule of M is prime. Then M is primely retractable if and only if $Hom(M, Rx) \neq 0$ for each $0 \neq x \in M$.

Corollary (4.5.4)

Every finitely presented prime module is primely retractable.

Proof:

Let M be a f.p. prime module. Let $0 \neq x \in M$, Then Rx is a direct summand of M[44, ,Exercies 32,p.163]. Therefore $Hom(M, Rx) \neq 0$ and according to (corollary (4.5.3)) M is primely retractable.

Corollary (4.5.5)

Every prime finitely generated projective module is primely retractable.

Proof:

Let M be a prime finitely generated projective module. Then M is finitely presented [44,Exercies 1,p.159] and by (corollary (4.5.4)) M is primely retractable.

In the following proposition we also give a sufficient condition for a module to by primely retractable.

Proposition (4.5.6)

Let M be a module such that every non-zero prime submodule of M contains a non-zero direct summand of M. Then M is primely retractable.

Proof:

Let N be a non-zero prime submodule of M. By hypothesis there is $0 \neq A \leq N$ and A is a direct summand of M. So, $M = A \oplus B$ for some $0 \neq B < M$. Let $\rho_A: M \to A$ be the projection homomorphism. Therefore $\rho_A \in Hom(M, A)$ and $i\rho_A \in Hom(M, N)$ where $i: A \to N$ is the inclusion homomorphism. If $i\rho_A = 0$, then $0 = i\rho_A(M) = \rho_A(M) \simeq A$ implies $A = 0$ which is a contradiction therefore $Hom(M, N) \neq 0$, hence M is primely retractable.

Now the following proposition gives a sufficient condition for a primely retractable module to be retractable:

Proposition (4.5.7)

Let M be a prime module such that every non-zero submodule of M contains a non-zero direct summand of M. if M is primely retractable, then M is retractable.

Proof:

Let $0 \neq N \leq M$. By hypothesis there is a direct summand of M, say K and $K \subseteq N$. Then K is prime in M [12,proposition 1.2,p.34] As M is primely retractable implies $Hom(M, K) \neq 0$ and hence $Hom(M, N) \neq 0$. Therefore M is retractable.

Corollary (4.5.8)

Let M be a prime module such that every non-zero submodule of M contains a non-zero direct summand of M. Then M is retractable if and only if M is primely retractable.

Proposition (4.5.9)

Let M be a finitely generated multiplication R-module. then M is primely retractable module.

Proof:

Let N be a non-zero prime submodule of M. Then $N = IM$ for some non-zero prime ideal I of R. [46]. But R is primely retractable by(Examples and Remarks1.2.2,(1)) implies that $Hom(R, I) \neq 0$. Let $0 \neq f: R \to I$ be a homomorphism. Let $f(1) = a$. Then $a \neq 0$. Define $g: M \to N$ by $g(M) = am$ for all $m \in M$ clearly, g is a well-defined homomorphism, $g \neq 0$ since M is faithful. Therefore $Hom(M, N) \neq 0$ this is what we wanted.

Corollary (4.5.10)

Every faithful cyclic R-module is also primely retractable.

Now, we introduce the concept of primely epi-retractable module as follows:

Definition (4.5.11)

A module M is called ***primely epi-retractable*** if every prime submodule of M is a homomorphic image of M. That is, whenever N is a prime submodule of M, then there exists an epimorphism from M onto N.

Examples and Remarkes (4.5.12)

(1) Every primely epi-retractable module is primely retractable.

(2) Q as a Z-module is primely epi-retractable.

(3) Z_{12} as a Z-module is primely epi-retractable module.

(4) Every semisimple module is primely epi-retractable.

(5) If M is a fully prime module, then M is primely epi-retractable if and only if M is epi-retractable.

Proposition (4.5.13)

Let M be a module satisfying (*). If M primely epi-retractable module, then every non-zero prime submodule of M is also primely epi-retractable.

Proof:

Let M *b*e a primely epi-retractable module and let N be a non-zero prime submodule of M. let K be a non-zero prime submodule of N. Then K is prime in M (since M satisfying (*)). Therefore there are epimorphisms $f: M \to N$ and $g: M \to K$. Define $h: N = f(M) \to K = g(M)$ by $h\big(f(m)\big) = g(m)$ for all $m \in M$. Clearly $h \in Hom(N, K)$. $h \neq 0$, for if $h = 0$. Then $h\big(f(M)\big) = 0 = g(M) = K$ which is a contradiction. Moreover h is an epimorphism, since $h(N) = h\big(f(M)\big) = g(M) = K$. Thus N is primely epi-retractable.

Corollary (4.5.14)

A direct summand of a prime and primely epi-retractable module is also primely epi-retractable.

roposition (4.5.15)

Let M be a primely epi-retractable module. Then *M/N* is primely epi-retractable.

Proof:

Let $\bar{0} \neq K/N$ be a prime submodule of M/N, where K is a proper submodule of M containing N properly. K/N is prime in M/N implies that K is prime in M [38,corollary 3.9]. Hence there is an epimorphism, say $f: M \to K$ (since M is primely epi-retractable by hypothesis). f induces a homomorphism $\bar{f}: M/N \to K/N$ with $\bar{f}(m+N) = f(m) + N$ for all $m \in M.\bar{f} \neq 0$, for if $\bar{f} = 0$, then $\bar{0} = \bar{f}(M/N)= f(M) + N = K + N$ (since f is an epimorphism). Hence $K + N = N$ implies $K = N$ which is a contradiction. Therefore $Hom(M/N, K/N)\neq 0$. Moreover $\bar{f}(M/N) = K/N$. Thus *M/N* is primely epi-retractable.

Lemma (4.5.16)

Let M_1 and M_2 be two R-module. If $N_1 \oplus N_2$ is a prime submodule of $M_1 \oplus M_2$. Then N_1 is a prime submodule of M_1 and N_2 is prime submodule of M_2.

Proof:

Let $r \in R$ and $x \in M_1$ such that $rx \in N_1$.Then $r(x,0) \in N_1 \oplus N_2$ So either$(x,0) \in N_{1\oplus}N_2$ or $r \in [N_1 \oplus N_2 : M_1 \oplus M_2]$ (since $N_1 \oplus N_2$is a prime submodule of $M_1 \oplus M_2$ by hypothesis). If $(x,0) \in N_1 \oplus N_2$ implies $x \in N_1$.If $r \in [N_1 \oplus N_2 : M_1 \oplus M_2]$,then $r(m_1, m_2) \in N_1 \oplus N_2$ for all $m_1 \in M_1$,for all $m_2 \in M_2$.Therefore $rm_1 \in N_1$ for all $m_1 \in M_1$,So $r \in [N_1 : M_1]$ and hence N_1 is prime submodule of M_1.similarly we prove that N_2 is prime submodule of M_2.

Proposition (4.5.17)

Let M_1 and M_2 be two primely epi-retractable modules such that $annM_1 + annM_2 = R$. Then $M_1 \oplus M_2$ is also primely epi-retractable.

Proof:

Let N be a non-zero prime submodule of $M_1 \oplus M_2$. Then $N = N_1 \oplus N_2$ for some submodule N_1 of M_1 and N_2 of M_2. By [31,proposition 4.2] and by Lemma (4.5.16) N_1 is prime in M_1 and N_2 is prime in M_2. Therefore there are epimorphisms $f: M_1 \to N_1$ and $g: M_2 \to N_2$. Define $h: M_1 \oplus M_2 \to N$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ for all $(m_1, m_2) \in M_1 \oplus M_2$. Clearly, h is a non-zero homomorphism and h is an epimorphism. Therefore $M_1 \oplus M_2$ is primely epi-retractable.

Corollary (4.5.18)

Let $\{M_i\}_{i=1}^{n}$ be a finite family of primely epi-retractable modules such that $\sum_{i=1}^{n} annM_i = R$. Then $\oplus_{i=1}^{n} M_i$ is also primely epi-retractable.

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Future Works

For future work the following problems could be recommended

1- Purely prime modules and purely prime submodules.
2- Purely Epi-retractable modules.
3- Generalized prime modules and Generalized prime submodules.
4- Primely uniform modules.
5- Primely Epi-retractable module.

المستخلص

لتكن R حلقة ابداليه ذات عنصر محايد، ولتكن M و N مقاسات يساريه احاديه على R. ولتكن $Hom_R(M,N)$ مجموعة كل التشاكلات المقاسية من M الى N. من المعروف ان خواص و تمييزات $Hom_R(M,N)$ كمقاس على R ممكن ان تحدد عن طريق خواص و تمييزات R، M و N وكذلك بعض خواص و تمييزات R، M و N ممكن ان تحدد عن طريق خواص و تمييزات$Hom_R(M,N)$، لذا فان العديد من الباحثين اهتموا بدراسة $Hom_R(M,N)$. بعض الدراسات تركزت حول استخدام خاصية $Hom_R(M,N) \neq 0$ لكل مقاس جزئي N غير صفري من M في هذه الحاله يطلق على M بانه مقاس المنكمشة، بينما اذا كان كل مقاس جزئي غير صفري من M يحوي على نسخة لـ M بمعنى انه يوجد تشاكل متباين في مجموعة $Hom_R(M,N)$ فيطلق على M بانه مقاس منضغط، من الواضح ان صنف المقاسات المنضغطه محتوات فعليا في صنف المقاسات المنكمشة.

في هذا العمل سوف نعطي دراسة مفصلة حول المقاسات المنضغطه الصغيرة والمقاسات المنكمشة الصغيرة، فضلاً على ذلك، اعمامات اخرى للمقاسات المنضغطه والمقاسات المنكمشة تم تقديمها ودراستها مثل المقاسات المنضغطه النقية والمقاسات المنكمشة النقيه، واخيراً المقاسات المنضغطه الاولية والمقاسات المنكمشة الاولية.